FAMAT Precalculus Study Guide

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Contents

1 Functions

1.1 Domain and Range

- The domain of $f(x)$ is the set of all real x such that $f(x)$ is defined.
- The range of $f(x)$ is the set of all possible values of $f(x)$ over the entire domain of $f(x)$

Common functions and their domain/range for $a, b > 0$ and integer k:

$f(x)$	Domain	Range
$mx + b$	$(-\infty, \infty)$	$(-\infty, \infty)$
$\frac{1}{x}$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$a(b)^x$	$(-\infty, \infty)$	$(0, \infty)$
$\log_a x$	$(0, \infty)$	$(-\infty, \infty)$
$\log_x a$	$(0, 1) \cup (1, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$\sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$\sin(x), \cos(x)$	$(-\infty, \infty)$	$[-1, 1]$
$\tan(x)$	$x \neq \frac{\pi}{2} + \pi k$	$(-\infty, \infty)$

1.2 Algebra of Functions

- Addition, Subtraction, Multiplication, and Division with functions behave the same way as it does with regular constants
- Don't forget to add like terms!
- Don't forget to simplify expressions!

Example: Let $f(x) = \frac{x-1}{x-2}$ $\quad g(x) = \frac{x}{x^2-3x+2}$ $\quad h(x) = \frac{1}{x-1}$

Simplify $f(x) + g(x) - h(x)$

$$
\begin{aligned}
& \frac{x-1}{x-2} + \frac{x}{x^2-3x+2} - \frac{1}{x-1} \\
= \quad & \frac{x^2-2x+1}{x^2-3x+2} + \frac{x}{x^2-3x+2} - \frac{x-3}{x^2-2x+3} \\
= \quad & \frac{x^2-2x+4}{x^2-3x+2}
\end{aligned}
$$

1.3 Composition of Functions

- When f is composed with g, or $(f \circ g)(x)$, this is the same thing as computing $f(g(x))$
- The domain of $(f \circ g)(x)$ is the intersection of the domain of $g(x)$ and $f(g(x))$

Example: Let $f(x) = \frac{x-1}{x-2}$ and $g(x) = \frac{x+1}{x-3}$
Compute $(f \circ g)(x)$ and its domain

$$\begin{aligned}
& (f \circ g)(x) \\
&= f(g(x)) \\
&= \frac{g(x) - 1}{g(x) - 2} \\
&= \frac{\frac{x+1}{x-3} - 1}{\frac{x+1}{x-3} - 2} \\
(f \circ g)(x) &= \frac{4}{-x + 7}
\end{aligned}$$

The domain of $g(x) = \frac{x+1}{x-3}$ is all reals except when $x - 3 = 0$ or when $x = 3$. The domain of $f(g(x)) = \frac{4}{-x+7}$ is all reals except when $-x + 7 = 0$ or when $x = 7$. So, the domain of $(f \circ g)(x)$ is all reals except $x = 3, 7$

1.4 Properties

- We must be able to determine if a function is symmetric about the axes, symmetric about origin, periodic, monotonic, bounded, or continuous.

Symmetry:
Compute $f(-x)$ and $-f(x)$.

- If $f(-x) = f(x)$, the function is *even* and therefore symmetric about the y-axis.
- If $-f(x) = f(x)$, the function is symmetric about the x-axis.
  - If the function is symmetric about the x-axis, then the only possible graphs it could be is $y = 0$ or points lying *exclusively* on the x axis
  - This is because a function must pass the vertical line test (to pass, all vertical lines must pass through the graph at most 1 point)
- If $f(-x) = -f(x)$, the function is *odd* and therefore symmetric about the origin.

Periodic:
Graph your function on the cartesian plane. If the graph repeats exact copies of itself over equal-length intervals from $(-\infty, \infty)$, then it is periodic.

- If your function is composed of trigonometric functions, chances are, it is periodic :)

Monotonic:
A monotonic function only moves in one direction (increasing or decreasing) across its entire domain

- Whenever $x_1 < x_2$ and $f(x_1) \leq f(x_2)$, then $f(x)$ is **monotonically increasing**
- Whenever $x_1 > x_2$ and $f(x_1) \geq f(x_2)$, then $f(x)$ is **monotonically decreasing**
- If $f(x)$ switches from increasing to decreasing (or vise versa) at any point, it is not a monotonic function
- **Example:** $\ln(x)$ is monotonically increasing

Bounded:
If $f(x)$ does not trend off to infinity in either direction, then it has both an upper and lower bound. In other words, it is *bounded*

- A function can also be only bounded above or bounded below. If this is the case, then it is not a bounded function.
- **Ex 1:** 2^x is bounded below by $y = 0$
- **Ex 2:** $-x^2 + 2$ is bounded above by $y = 2$
- **Ex 3:** $2\sin(3x)$ is bounded because it is bounded above by $y = 2$ *and* below by $y = -2$

Continuous:
Three conditions must be met for $f(x)$ to be continuous at $x = a$

1. $f(a)$ is defined
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

A function is continuous along its domain if all points in the domain satisfy the above 3 conditions

- **Ex 1:** $\log(x)$ is a continuous function
- **Ex 2:** $\frac{|x-1|}{1-x}$ is not a continuous function because there is a "jump" at $x = 1$
- **Ex 3:** $\tan(x)$ is not a continuous function because there are infinitely many asymptotes

1.5 Points of discontinuity

In a function, usually a rational or piecewise function, the graph will demonstrate points of removable discontinuities (holes) and/or non-removable discontinuity

Removable Discontinuity:
Even though the limit exists, a function is not defined at a point of removable discontinuity. In rational functions, this usually occurs when there are common factors that cancel out in both the numerator and denominator. To find the coordinates of the hole, cancel out the common factor and plug in that value of x that makes the common factor equal to 0 into the simplified function.

Ex 1: At what value of x does there exist a point of removable discontinuity in the function $f(x) = \frac{x^2-1}{x+1}$
Solution: The numerator can be factored to $(x+1)(x-1)$. Both the numerator and denominator share the $x+1$ factor. Setting this binomial equal to 0 gives us an x value of $\boxed{\text{-1}}$. To find the coordinates, plug in $x = -1$ into the simplified expression to get $(-1, -2)$

Ex 2: Find all points of removable discontinuity in of the function $\frac{x^3-2x^2-x+2}{x^2-x-2}$
Solution: The first step is always to factor. We can factor the expression as $\frac{(x-2)(x-1)(x+1)}{(x-2)(x+1)}$. The common factors are $(x+1)$ and $(x-2)$. They are equal to 0 when $x = -1, 2$. The simplified expression is $x - 1$. So, the coordinates of the removable discontinuities are $\boxed{(-1, -2) \text{ and } (2, 1)}$.

Ex 3: Find the value of a such that the following function has a point of removable discontinuity: $f(x) = \begin{cases} -x^2+1 & (x < 2) \\ -2 & (x = 2) \\ ax+4 & (x > 2) \end{cases}$
Solution: We notice that there is a jump at $x = 2$. It goes from $-(2)^2 + 1 = -3$ from the left hand side, but at $x = 2$, it is defined to be -2. In order for there to be a hole, the right hand side of the function must also approach -3 as $ax + 4$ goes to $x = 2$. We can set up the equation $-3 = 2a + 4$ to get $\boxed{a = -3.5}$

Non-Removable Discontinuity:
At a point of non-removable discontinuity, the limit as $f(x)$ approaches from both sides does NOT exist. This is usually the location of an asymptote or a jump.

- **Ex 1:** $\frac{|x-1|}{1-x}$ shows a point of non-removable discontinuity because of the jump at $x = 1$
- **Ex 2:** $\tan(x)$ shows a point of non-removable discontinuity at $x = \frac{\pi}{2}$ because there is a vertical asympotote there.

1.6 Asymptotes & Intercepts

Many questions will give you a rational or rational-like function and ask to find all asymptotes of the expression. Below are general strategies you should take when approaching these types of questions.

- Find the limit of the function as x approaches $+\infty$ (gives a horizontal asymptote)
- Find the limit of the function as x approaches $-\infty$ (gives horizontal asymptote)
- Find values of x that yield a denominator of 0 (gives vertical asymptote)

Intercepts

- To find the x intercept of a function, simply set the function equal to 0 and solve for x
- To find the y intercept of a function, just plug in $x = 0$ and look at the value
- $\uparrow$ Doing this should not be that deep. Quick and EZ points right here.

Ex 1: Find all asymptotes of $\frac{x-5}{10-|x|}$
Solution: As x approaches $+\infty$, the function becomes $\frac{x}{-x} = -1$. As x approaches negative infinity, the function becomes $\frac{x}{x} = 1$. The denominator is 0 when $x = \pm 10$. So, our asymptotes are $x = \pm 10$ and $y = \pm 1$.

Ex 2: Find all asymptotes of $\frac{x^2-1}{2x^2+x-1}$
Solution: Rewrite the function as $\frac{(x-1)(x+1)}{(2x-1)(x+1)}$. When solving for asymptotes of a rational function, we can cancel out common factors as if they weren't there. This gives $\frac{x-1}{2x-1}$. As this function approaches $\pm\infty$, it yields $\frac{1}{2}$. When $x = \frac{1}{2}$, the denominator is 0. So, our asymptotes are $y = \frac{1}{2}$ and $x = \frac{1}{2}$.

Ex 3: Find all asymptotes of $\frac{2e^{-x}+5x}{e^{-x}+2x}$
Solution: As x approaches $+\infty$, the e^{-x} terms approach 0, leaving us with $\frac{5x}{2x} = \frac{5}{2}$. As x approaches $-\infty$, the e^{-x} terms become large compared to the linear terms, giving $\frac{2e^{-x}}{e^{-x}} = 2$. It is possible for the denominator to be 0, but we cannot solve for the exact values using algebra. This question will likely be formatted as "select from I, II, III....". This means we should choose the options containing $y = 2, \frac{5}{2}$.

1.7 Slopes

In calculus, you will explore the concept of derivatives in much more detail than you do in Precalc. In Precalculus, questions will sometimes give you the equation of a function $f(x)$ and the equation of its derivative $f'(x)$. The derivative of a function $f'(c)$ gives the slope of $f(x)$ at $x = c$.
Example: Let $f(x) = 2x^2 - 4x$. If the slope of the function can be described as $f'(x) = 4x - 4$, find the equation of the line tangent to $f(x)$ at $x = 5$
Solution: First we find the slope of $f(x)$ at $x = 5$. This is $f'(x) = 4(5) - 4 = 16$. To find

the equation of a line given its slope, we also need a point. At $x = 5$, $f(5) = 30$. So, we have $30 = 16(5) + b$, where $b = -50$. The equation of the tangent line to $2x^2 - 4x$ at $x = 5$ is $\boxed{y = 16x - 50}$

1.8 Parametrics

- We can create a graph in the cartesian plane by expressing x and y relationships in terms of a *parameter*, usually t.
- The strat is to rewrite the expression by eliminating the parameter.
- To do this, solve for the parameter in terms of x or y and then substitute that into the other relationship.
- Be on the lookout for trig identity manipulation!

Ex 1: $f(x) = \begin{cases} x = t^2 + 1 \\ y = \sqrt{t} \end{cases}$

Solution: Square the second equation to get $t = y^2$. Plug this into the first equation to get $x = y^4 + 1$. This is the top half of a sideways parabola-like graph because negative values of y do not exist ($\sqrt{t}$ is never negative).

Ex 2: Find the area enclosed by $f(x) = \begin{cases} x = 2\sin(t) + 1 \\ y = -\cos(t) + 2 \end{cases}$

Solution: Isolate each trig expression to get $\sin(t) = \frac{x-1}{2}$ and $\cos(t) = -y + 2$. Square both equations and use the trig identity: $\cos^2(t) + \sin^2(t) = 1$ to get

$$\frac{(y-2)^2}{1^2} + \frac{(x-1)^2}{2^2} = 1$$

Using area of an ellipse formula, we get $A = \pi(1)(2) = \boxed{2\pi}$.

1.9 Inverses

- The inverse of a function essentially swaps the values of x and $f(x)$.
  - A consequence of this property is that a function is symmetric to its inverse along the line $y = x$
  - This means that the intersection(s) of a function and its inverse lies on $y = x$. To solve, set $f(x) = x$. Useful when asked to compute for point(s) of intersection!
- We can solve for a function's inverse by swapping x and y and resolve for y.
- If $g(x)$ is the inverse of $f(x)$, then the domain of $f(x)$ is the range of $g(x)$
- The range of $f(x)$ is the domain of $g(x)$.

– This is useful when asked to solve for the range of a function's inverse. We can simply solve for the domain of the function.

- If the function passes the horizontal line test (All horizontal lines drawn never intersects a function more than once), the function's inverse is also a function.

Ex 1: Let $y = x^2 + 2x - 3$. What is the point(s) of intersection of y and its inverse?
Solution: Set $x^2 + 2x - 3 = x$ and solve. We get $x = \frac{-1\pm\sqrt{13}}{2}$. Solution is $\left(\frac{-1\pm\sqrt{13}}{2}, \frac{-1\pm\sqrt{13}}{2}\right)$.

Ex 2: Let $f(x) = \frac{x^2-2x-1}{x^2-2x+1}$. What is the range of $f(x)$?
Solution: Solve for the inverse function and then compute its domain. The resulting interval is also the range of $f(x)$. Swapping x and y gets

$$x = \frac{y^2 - 2y - 1}{y^2 - 2y + 1}$$

Isolating 0 gives

$$y^2(x-1) + y(-2x+2) + x + 1 = 0$$

Use quadratic formula to solve for y:

$$y = \frac{2x - 2 \pm \sqrt{-8x+8}}{2x-2}$$

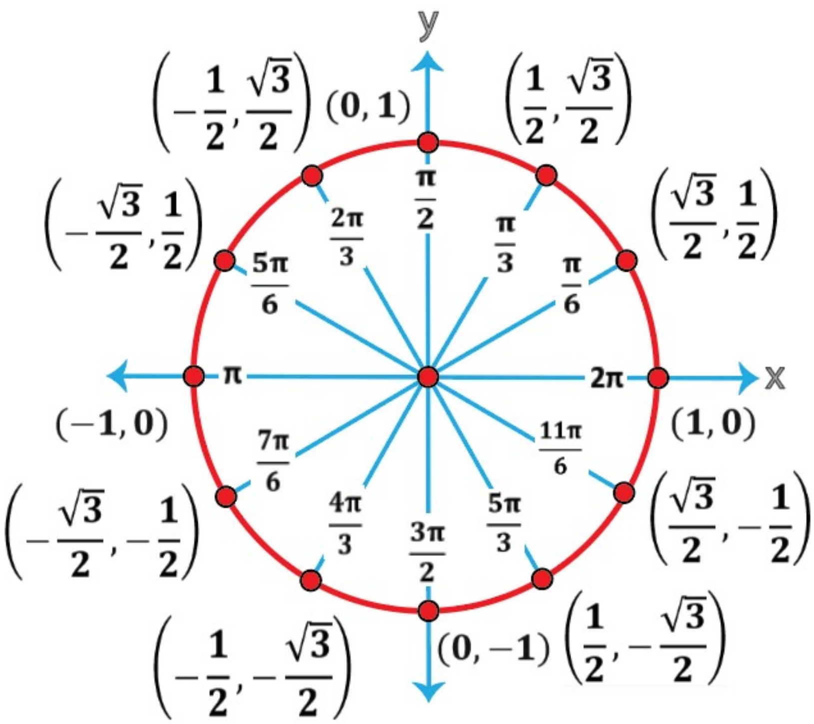
The domain of this function is $x < 1$. Therefore the range of $f(x)$ is $\boxed{(-\infty, 1)}$

2 Trig Functions

2.1 Unit circle

- We must be able to compute trigonometric expressions (Ex: $\sin(\frac{\pi}{6}) = \frac{1}{2}$) and their inverses (Ex: $\tan^{-1}(-\frac{\sqrt{3}}{3}) = -\frac{\pi}{6}$)

- We can use a unit circle to help us with computation



- Let x be the x coordinate and y be the y coordinate of the intersection between a radius drawn θ degrees counterclockwise from the horizontal and a **unit circle**.

Function	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
Value	y	x	$\frac{y}{x}$	$\frac{1}{y}$	$\frac{1}{x}$	$\frac{x}{y}$

Here is a list of common values for the 6 trig functions

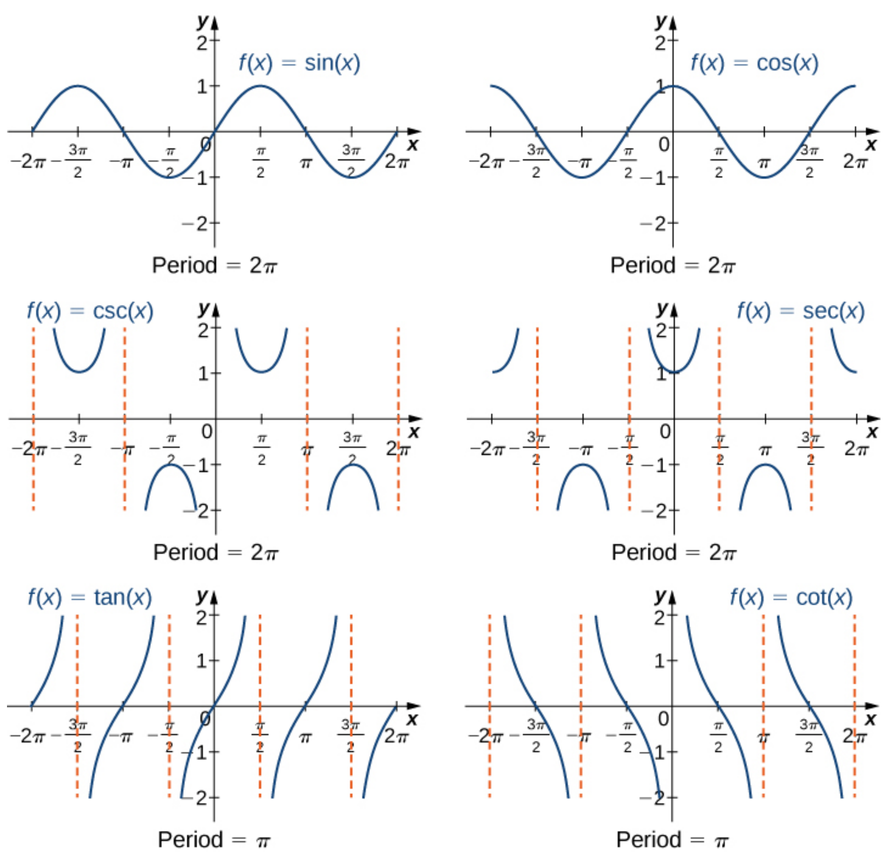
θ		*sin* θ	*cos* θ	*tan* θ	*csc* θ	*sec* θ	*cot* θ
Rad	**Deg**						
0 / 2π	0	0	1	0	Undef	1	Undef
$\pi/6$	30	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
$\pi/4$	45	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	60	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
$\pi/2$	90	1	0	Undef	1	Undef	0
$2\pi/3$	120	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2	$-\sqrt{3}/3$
$3\pi/4$	135	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
$5\pi/6$	150	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$	$-\sqrt{3}$
π	180	0	-1	0	Undef	-1	Undef
$7\pi/6$	210	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$	$\sqrt{3}$
$5\pi/4$	225	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
$4\pi/3$	240	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2	$\sqrt{3}/3$
$3\pi/2$	270	-1	0	Undef	-1	Undef	0
$5\pi/3$	300	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2	$-\sqrt{3}/3$
$7\pi/4$	315	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
$11\pi/6$	330	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$	$-\sqrt{3}$

2.2 Functions

- Given a trigonometric function in the form $a\sin(bx+c)+d$ (or cos), we must be able to compute the period, amplitude, phase shift, and vertical shift
  - **Period:** Always equal to $\frac{2\pi}{b}$
  - **Amplitude:** Always equal to a
  - **Phase shift:** First compute $-\frac{c}{b}$ (from setting $bx+c=0$). The phase shift is equal to $\phi = -\frac{c}{b}+\frac{2\pi}{b}k$ for integer k, where ϕ is as close to 0 as possible.
    * Finding phase shift is very disputable as some test writers may only accept $-\frac{c}{b}$.
  - **Vertical shift:** Always equal to d
- To graph a function in the above form, first plug in $-\frac{c}{b}$. This point is a relative max/min of your function. From here, plug in $-\frac{c}{b}$ plus multiples of one fourth the value of your period. The resulting values are also relative max/min. Play connect the dots to create a sinusoidal graph.

- Given a trigonometric function in the form $a\tan(bx+c)+d$ (or cot), we must be able to compute the period, amplitude, phase shift, and vertical shift
  - **Period:** Always equal to $\frac{\pi}{b}$
  - **Amplitude:** Does not exist (∞)
  - **Phase shift:** First compute $-\frac{c}{b}$ (from setting $bx+c=0$). The phase shift is equal to $\phi = -\frac{c}{b}+\frac{\pi}{b}k$ for integer k, where ϕ is as close to 0 as possible.
    * Finding phase shift is very disputable as some test writers may only accept $-\frac{c}{b}$.
  - **Vertical shift:** Always equal to d
- To graph a function in the above form, first solve for a value of x and plot the corresponding point that makes the tan/cot component equal to 0.
- Next, along the horizontal line containing the above point, plot points that are equally spaced a distance of one period away from each other ($\frac{\pi}{b}$).
- Next, draw dotted lines along the perpendicular bisector of each pair of consecutive points to represent the asymptotes.
- Draw repeating curves going through each of the plotted points from $-\infty$ to $+\infty$, depending on the sign of a and if the function is cot or tan. (Plug in test values to determine this).

- Given a trigonometric function in the form $a\sec(bx+c)+d$ (or csc), we must be able to compute the period, amplitude, phase shift, and vertical shift
  - **Period:** Always equal to $\frac{2\pi}{b}$
  - **Amplitude:** Does not exist (∞)
  - **Phase shift:** First compute $-\frac{c}{b}$ (from setting $bx+c=0$). The phase shift is equal to $\phi = -\frac{c}{b} + \frac{2\pi}{b}k$ for integer k, where ϕ is as close to 0 as possible.
    - Finding phase shift is very disputable as some test writers may only accept $-\frac{c}{b}$.
  - **Vertical shift:** Always equal to d
- To graph a function in the above form, first solve for a value of x and plot the corresponding point that makes the sec/csc component equal to 0.
- Next, along the horizontal line containing the above point, plot points that are equally spaced a distance of one period away from each other ($\frac{2\pi}{b}$).
- Next, draw dotted lines along the perpendicular bisector of each pair of consecutive points to represent the asymptotes.
- Draw 1 parabola-like curve between each asymptote with the "vertex" as the plotted points above. Alternate between opening up or down, depending on the sign of a and if the function is sec/csc (plug in test values to determine this).



3 Trig Identities

3.1 Basics

- $\cos x$ is an even function
  1. $\cos x = \cos(-x)$
  2. $\cos x = -\cos(\pi - x)$
- $\sin x$ is an odd function
  1. $\sin x = -\sin(-x)$
  2. $\sin x = \sin(\pi - x)$
- $\tan x$ is an odd function
  1. $\tan x = -\tan(-x)$
  2. $\tan x = tan(\pi + x)$
- $\cos(x) = \sin(\frac{\pi}{2} - x)$

3.2 Pythagorean Theorem Derived Identities

- A point on the unit circle is expressed as $(\cos\theta, \sin\theta)$
- Using Pythagorean Theorem, we get identity 1:

$$\cos^2\theta + \sin^2\theta = 1$$

- From here, we can derive 2 other identities
  - Divide both sides of identity 1 by $\cos^2\theta$ to get:

$$1 + \tan^2\theta = \sec^2\theta$$

  - Divide both sides of identity 1 by $\sin^2\theta$ to get:

$$\cot^2\theta + 1 = \csc^2\theta$$

- **Ex 1:** Solve $2\sin^2 x + 3\cos x = 0$ over $0 \leq x \leq 2\pi$
  Solution: Using identity 1, we get $\sin^2 x = 1 - \cos^2 x$. We can substitute this into the equation to get:

$$2(1 - \cos^2 x) + 3\cos x = 0$$

  Expanding and multiplying by -1 gives:

$$2\cos^2 x - 3\cos x - 2 = (2\cos x + 1)(\cos x - 2) = 0$$

  $\cos x \neq 2$ because cosine is never greater than 1, so $\cos x = -\frac{1}{2}$. This gives $\boxed{x = \frac{2\pi}{3}, \frac{4\pi}{3}}$

- **Ex 2:** Solve $(\sec x + \tan x)^2 = 3$ over $0 \leq x \leq 2\pi$
Solution: Taking the square root of both sides gives $\sec x + \tan x = \sqrt{3}$ or $-\sqrt{3}$. Recall that $1 + \tan^2\theta = \sec^2\theta$, or $\sec^2\theta - \tan^2\theta = (\sec x + \tan x)(\sec x - \tan x) = 1$. After we substitute $\sec x + \tan x = \pm\sqrt{3}$, we find that $\sec x - \tan x = \pm\frac{\sqrt{3}}{3}$. We have 2 systems of equations. We have that $2\sec x = \frac{4\sqrt{3}}{3}$ or $2\sec x = -\frac{4\sqrt{3}}{3}$. Solving for x gives $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$. However, because the original equation is squared, we must be wary of extraneous solutions. After testing all 4 values, we find that only $\boxed{x = \frac{\pi}{6}, \frac{5\pi}{6}}$.

3.3 Sum & Product of Angles

- Test writers will ask many questions on angle addition. They expect you to know the formulas below:
  1. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
  2. $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
  3. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
  4. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
  5. $\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
  6. $\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$
- By manipulating the above identities, we can use **sum to product** and, likewise, product to sum formulas
  1. $\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$
  2. $\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$
  3. $\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$
  4. $\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$

3.4 Double Angle

- Using equations 1, 3, and 5 of the angle addition section, we can derive the following double angle formulas:
  1. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
  2. $\sin(2\theta) = 2\cos\theta\sin\theta$
  3. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$
- **Example (Common Question):** Simplify $(\cos 20°)(\cos 40°)(\cos 80°)$
Solution: Whenever you see a lot of double-angle looking trig functions multiplied together, you want to exploit double angle. Multiply the top and bottom of the expression by $\sin 20°$:
$$\frac{(\sin \mathbf{20°})(\cos \mathbf{20°})(\cos 40°)(\cos 80°)}{\sin 20°}$$

By double angle, $(\sin 20°)(\cos 20°) = \frac{\sin 40°}{2}$. Substituting:

$$\frac{(\sin 40°)(\cos 40°)(\cos 80°)}{2\sin 20°}$$

Combining:

$$\frac{(\sin 80°)(\cos 80°)}{4\sin 20°}$$

Combining:

$$\frac{(\sin 160°)}{8\sin 20°}$$

Because $\sin 160° = \sin 20°$ (to know why, see Basics), we can simplify the expression to $\boxed{\frac{1}{8}}$

3.5 Half Angle

Test writers may ask questions on half angle. By using the first two product to sum formulas, we can find half angle for sin and cos:

- $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}$
- $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$
  - The above two formulas do NOT mean plus AND minus. It is plus OR minus. In order to determine the sign, determine what quadrant $\frac{\theta}{2}$ lies in.
- $\tan\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$

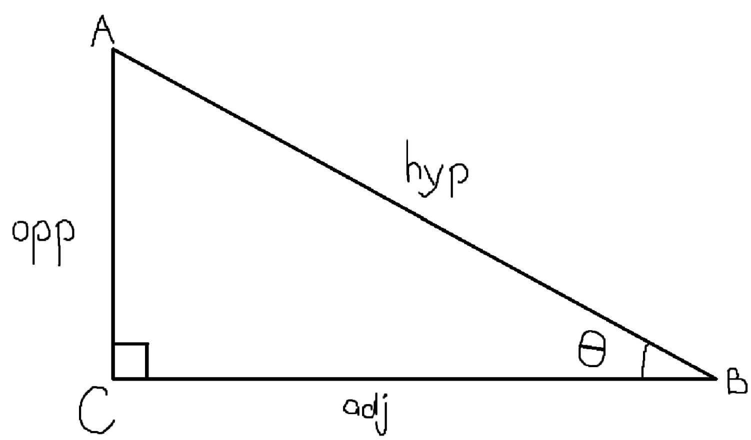
Random exercise: Use your trig identities to simplify $\sin 15°$ (aka $\sin\left(\frac{\pi}{12}\right)$)

Hint: $\frac{\pi}{12} = \frac{\frac{\pi}{6}}{2} = \frac{\pi}{3} - \frac{\pi}{4}$

4 Trig with Triangles

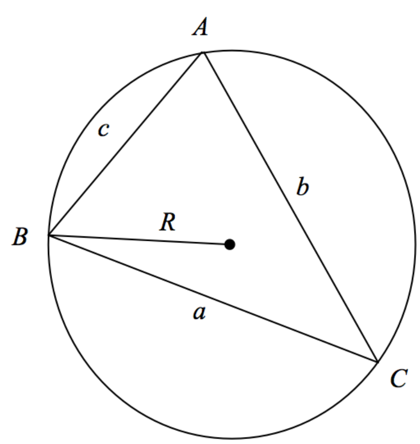
4.1 Sohcahtoa

- **Solve a right triangle** given two sides, or a side and an acute angle.
  - Use $\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$, $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$, $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$.
  - Right triangles: SOH–CAH–TOA
    - **S**in-**O**pposite-**H**ypotenuse
    - **C**os-**A**djacent-**H**ypotenuse
    - **T**an-**O**pposite-**A**djacent



4.2 Law of Sines

- Law of sines is useful when you are given 2 angles and a corresponding side length



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

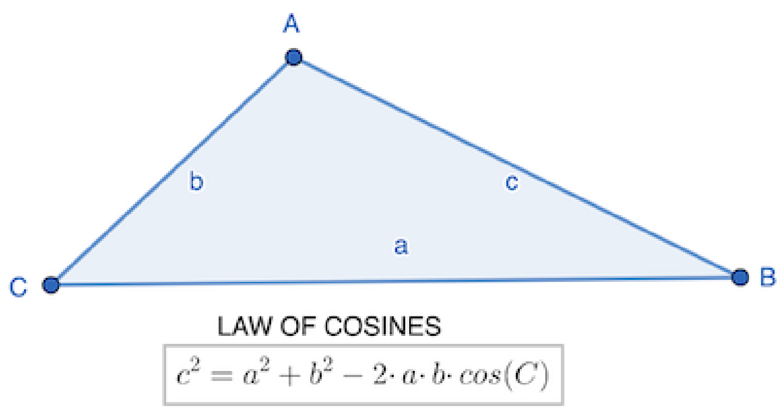
- $R =$ Triangle's circumradius

4.3 Law of Cosines

Use Law of Cosines when:

- you are given two side lengths and the included angle
- you are given all three sides and asked to find an angle

$$c^2 = a^2 + b^2 - 2ab\cos C, \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



- Note that you can label the A, B, and C vertices of a triangle however you'd like. It does not have to be in the orientation as in the picture shown above.

4.4 Area

- Finding the area of a right triangle is easy. However, in Precalc, test writers will often ask to find the area of an oblique triangle.
- When given *3 side lengths* of length a, b, c, use **Heron's Formula**:

$$A = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \tfrac{a+b+c}{2}$$

- When given *2 side lengths* of length a, b and their *included angle* of measure C, use **this** (idk the name):

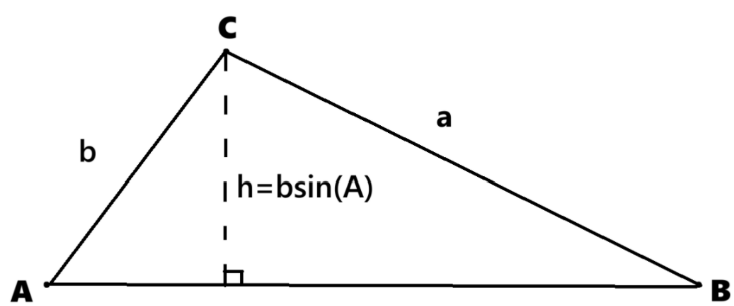
$$A = \frac{1}{2}ab\sin C$$

Ex 1: A triangle has side lengths 7, 15, and 20. Find the length of the longest altitude.
Solution: The longest altitude will have its foot lying on the shortest side length of 7. If we find the area of the triangle, we can use then use $A = \frac{1}{2} \cdot 7 \cdot h$, where h is the length of the altitude. Using Heron's, we find $s = 21$. We get $A = \sqrt{21 \cdot 14 \cdot 6 \cdot 1} = 42$. Plugging in $A = 42$ into our previous formula gives us $\boxed{h = 12}$

Ex 2: The longest sides of a triangle have lengths of 6 and 7. What is the maximum possible area of this triangle?
Solution: Call the included angle of these two side lengths to be C. The area is $A = \frac{1}{2} \cdot 6 \cdot 7 \cdot \sin C = 21 \sin C$. The maximum value of $\sin C$ is 1, so the maximum area of this triangle is $\boxed{21}$. This is a right triangle!

4.5 SSA Existence

I think this topic is kind of annoying...



- Given A (acute), side a opposite A, and side b adjacent A. Let $h = b \sin A$.
  - If $a < h$: **0 triangles.**
  - If $a = h$: **1 right triangle.**
  - If $h < a < b$: **2 triangles**
  - If $a \geq b$: **1 triangle.**
- If $A > 90°$: a triangle exists only if $a > b$ (then **1 triangle**).

5 Conic Sections & Loci

5.1 Lines

- Standard form: $Ax + By = C$
  - Slope: $-\frac{A}{B}$
  - x-intercept: $\frac{C}{A}$
  - y-intercept: $\frac{C}{B}$
- Slope Intercept form: $y = mx + b$
  - Slope: m
  - x-intercept: $-\frac{b}{m}$
  - y-intercept: b
- Polar form: $r = \frac{1}{A\cos\theta + B\sin\theta}$
  - Slope: $-\frac{A}{B}$
  - x-intercept: $\frac{1}{A}$
  - y-intercept: $\frac{1}{B}$
  - You may find lines in this form awkward to work with, so I recommend multiplying by the denominator to convert the equation to rectangular form (you get $Ax + By = 1$).
- Extra:
  - Distance from a point (a, b) to the line $Ax + By = C$ is

$$D = \frac{Aa + Bb - C}{\sqrt{A^2 + B^2}}$$

  - Be careful not to mix up the captial and lower case variables!
  - Make sure your equation of the line has C on the RHS and not the LHS!

5.2 Circles

Definition: The locus of points in 2D space equidistant from a specific point (h, k) at a constant distance r.

- Rectangular equation: $(x - h)^2 + (y - k)^2 = r^2$
- Center (h, k), radius r
- Polar equation: $r =$ insert length of radius

- Alternate polar equation: $r = k\cos\theta$
  Extra:

- 3 non-collinear points determine a circle
  - To find center, select 2 pairs of points. Compute the perpendicular bisector of each pair. Then, compute the point of intersection of the perpendicular bisectors. That is the center.
  - Once center is found, you can compute radius by using distance formula between center and a given point.

Example: A triangle has vertices at $(5, 9), (1, 7), (8, 0)$. Compute its circumradius.
Solution:

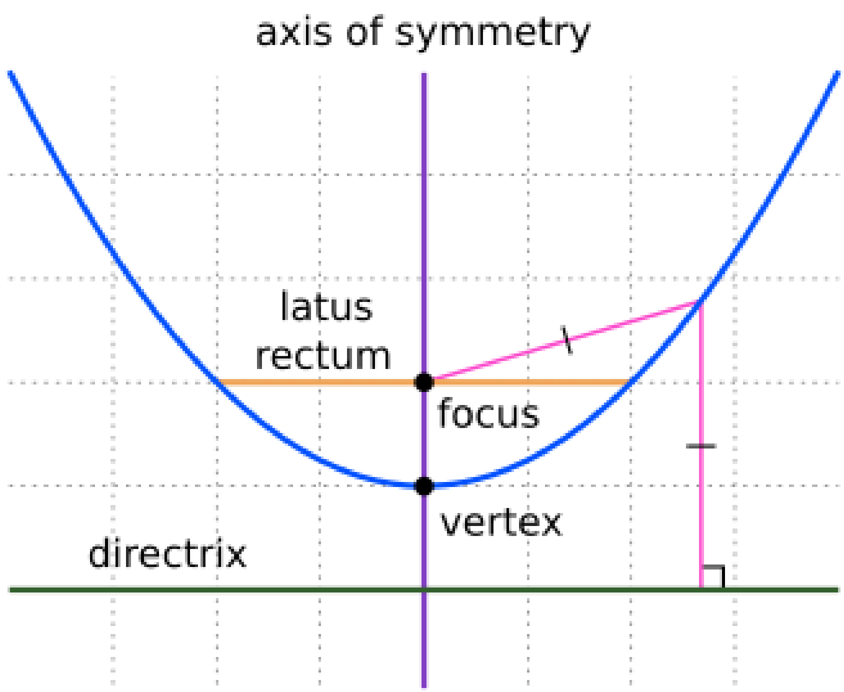
1. Find the perpendicular bisector between the first two points. Midpoint is $(3, 8)$. Slope of line is $\frac{1}{2}$ so p.b. slope is -2. Use $y = -2x + b$ to find a p.b. of $y = -2x + 14$.
2. Find the perpendicular bisector between the last two points. Midpoint is $(\frac{9}{2}, \frac{7}{2})$. Slope of line is -1 so p.b. slope is 1. Use $y = 1x + b$ to find a p.b. of $y = x - 1$.
3. Set p.b.'s equal to each other: $-2x + 14 = x - 1$. We get $x = 5, y = 4$.
4. Center is $(5, 4)$. Distance between the center and any one of the triangle's vertices is $\boxed{5}$.

5.3 Parabolas

Definition: The locus of points in 2D space equidistance from a point (the focus) and a line (the directrix).

- Quadratic form: $y = ax^2 + bx + c$
  - Useful when they straight up give you 3 points that fit a quadratic model.
  - Once you solve for a, b, c, convert equation to either vertex or focus-directrix form to extract information more easily
- Vertex form: $y = a(x - h)^2 + k$
  - Useful when they give you the coordinates of the vertex and another point.
  - Vertex: (h, k)
  - Opens up if $a > 0$, opens down if $a < 0$
- Focus-Directrix form: $4p(y - k) = (x - h)^2$ or $4p(x - h) = (y - k)^2$
  - Useful when they give you graphical information of focus or a vertex
  - If x is squared, it is an up-down parabola
    - p is negative if it opens down

    - p is positive if it opens up
  - If y is squared, it is a left-right parabola
    - p is negative if it opens left
    - p is positive if it opens right
  - Latus rectum is a parabolic chord that is perpendicular to axis of symmetry and goes through the focus
  - Length of latus rectum: $4p$
  - Distance from vertex to focus or vertex to directrix: p
  - Vertex: (h, k)



- Polar form: $r = \frac{p}{1 \pm \cos\theta}$
  - Only use this form if the question gives you information in polar plane.
  - Origin is focus
  - Distance from vertex to focus: $\frac{1}{2}p$
- Extra:
  - A parabola's eccentricity is 1
  - Eccentricity of a conic is defined to be the distance of a point on a conic to a focus divided by the distance of the same point on the conic to the corresponding directrix.

Ex 1: Find the equation of the parabola with a vertical axis of symmetry going through the origin, (2, -2), and $(4, 4)$

Solution: Let $f(x) = ax^2 + bx + c$. Set up a system of equations:

$$\begin{aligned} 0 &= c \\ -2 &= 4a + 2b + c \\ 4 &= 16a + 4b + c \end{aligned}$$

Solving this system gets $a = 1$, $b = -3$, and $c = 0$. The equation is $\boxed{x^2 - 3x}$

Ex 2: Isa throws a tantrum at $t = 0$ starting an an initial height of 0 meters. It follows a parabolic path. If the tantrum reaches a maximum height of 18 meters after 3 seconds, how high is his tantrum after 4 seconds?

Solution: We are given a vertex in disguise, which is $(3, 18)$. We can write the equation as $y = a(t-3)^2 + 18$. Since $y = 0$ at $t = 0$, we can say $0 = 9a + 18$, or $a = -2$. The equation is therefore $y = -2(t-3)^2 + 18$. At $t = 4$ seconds, Isa's tantrum is $\boxed{16}$ meters high.

Ex 3: A parabola opening to the left has a latus rectum of length 10 and a vertex at $(10, 4)$. What is its largest y intercept?

Solution: We have information on the latus rectum, and since it opens left, we use $4p(x-h) = (y-k)^2$. We know $|4p| = 10$, and since it opens left, p must be $-\frac{5}{2}$, which means $4p = -10$. Therefore, the equation is $-10(x-10) = (y-4)^2$. At $x = 0$, the y intercepts are 14 and -6. The largest is $\boxed{14}$.

Ex 4: Find the equation of a parabola with vertex at the origin and directrix $2x + y = 1$.

Solution: We use the fact that the eccentricity of a parabola is 1. Let's call a point on this parabola to be (x, y). The distance from (x, y) to the focus divided by the distance from (x, y) to the directrix is 1. In other words, these two distances are the same.

$$\text{Distance to focus: } \sqrt{x^2 + y^2}$$

$$\text{Distance to directrix: } \frac{2x + y - 1}{\sqrt{2^2 + 1^2}}$$

Setting them equal and simplifying:

$$\begin{aligned} \sqrt{x^2 + y^2} &= \frac{2x + y - 1}{\sqrt{2^2 + 1^2}} \\ \sqrt{5x^2 + 5y^2} &= 2x + y - 1 \\ 5x^2 + 5y^2 &= 4x^2 + y^2 + 1 + 4xy - 2y - 4x \end{aligned}$$

Isolating 0, we get an equation of $\boxed{x^2 - 4xy + 4y^2 + 4x + 2y - 1 = 0}$

5.4 Ellipses

Definition: The locus of points in a 2D plane such that the sum of the distances from two fixed points (foci) is constant

- Standard form (horizontal major axis):

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad a > b > 0$$

  - Center (h, k)
  - Vertices, Endpoints of major axis: $(h \pm a, k)$
  - Co-vertices, Endpoint of minor axis: $(h, k \pm b)$
  - Foci: $(h \pm c, k)$ with $c^2 = a^2 - b^2$
  - Latus Rectum length: $\frac{2b^2}{a}$
  - The latera recta are two elliptical chords parallel to the minor axis and going through the foci
  - Vertical ellipse: swap x/y roles
  - Eccentricity: $e = \frac{c}{a}$ $(0 < e < 1)$
  - Constant distance in definition: $2a$
  - The two directrices are parallel to the minor axis lying a distance of $\frac{a^2}{c}$ away from the center

- Polar form (horizontal major axis):

$$r = \frac{ep}{1 \pm e\cos\theta}$$

  - Origin is location of one of the foci
  - I recommend plugging in quadrantal angles and plot the vertices and co-vertices on the Cartesian plane. Then, convert information into rectangular.
  - p is distance from a focus to origin

Ex 1: A tunnel has an elliptical arch that is 18 feet high and 36 feet across. If a truck is 12 feet wide, what is the maximum height it can be to pass through safely?
Solution: This is a horizontal ellipse with $a = 18$ and $b = 9$. We can center the ellipse at (0,0). The equation is $\frac{x^2}{18^2} + \frac{y^2}{9^2} = 1$. If the truck is 12 feet wide, the top edges lie at $x = 6$. Plugging this into the equation gives $\boxed{y = 6\sqrt{2}}$.
Ex 2: An ellipse is described by the equation $\frac{x^2}{4^2} + \frac{y^2}{8^2} = 1$. What is the area of the rectangle with vertices as the endpoints of the latera recta?
Solution: The length of this rectangle is the distance between the two foci, or $2c$. The height is the length of the latus rectum, or $\frac{2b^2}{a}$. The area is $\frac{4b^2c}{a}$. Knowing that $a = 8$ and $b = 4$, we find that $c = 4\sqrt{3}$. Therefore, the area is $\boxed{32\sqrt{3}}$.

5.5 Hyperbolas

Definition: The locus of points in a 2D plane such that the absolute value of the differences of the distances from two fixed points (foci) is constant

- Standard form (horizontal transverse axis):

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- Standard form (vertical transverse axis):

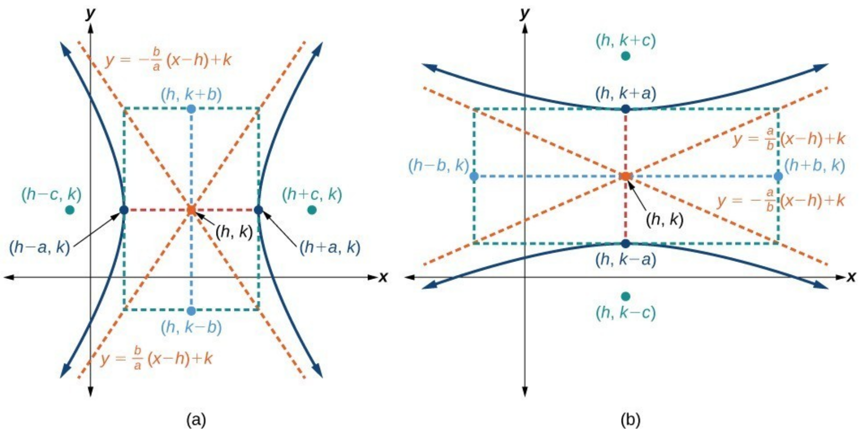
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

  - Center: (h, k)
  - Length of Transverse Axis: $2a$
  - Length of Conjugate Axis: $2b$
  - $a^2 + b^2 = c^2$
  - Latus Rectum length: $\frac{2b^2}{a}$
  - The latera recta are two chords parallel to the conjugate axis that go through the foci
  - Asymptotes (horizontal): $y - k = \pm\frac{b}{a}(x-h)$
  - Asymptotes (vertical): $y - k = \pm\frac{a}{b}(x-h)$
  - Eccentricity: $e = \frac{c}{a} > 1$
  - Foci lie along transverse axis a distance c away from center
  - The two directrices are parallel to conjugate axis and lie a distance of $\frac{a^2}{c}$ away from the center

- Polar form (horizontal major axis):

$$r = \frac{ep}{1 \pm e\cos\theta}$$

  - $e > 1$
  - Origin is location of one of the foci
  - (Same as ellipse) I recommend plugging in quadrantal angles and plot the vertices and co-vertices on the Cartesian plane. Then, convert information into rectangular.
  - p is distance from a focus to origin



5.6 General Quadratic Form

Any rotated or unrotated conic can be written in the form $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

- **Discriminant test:**
  - If $A = C$ and $A > 0$, conic is circle
  - If $B^2 - 4AC = 0$, conic is parabola
  - If $B^2 - 4AC < 0$, conic is ellipse
  - If $B^2 - 4AC > 0$, conic is hyperbola
- **¡WARNING!**
  - Some test writers like to be trappy and make the conic degenerate
  - To be safe, perform the degeneracy test by evaluating the matrix:

$$M = \begin{vmatrix} A & \frac{B}{2} & \frac{D}{2} \\ \frac{B}{2} & C & \frac{E}{2} \\ \frac{D}{2} & \frac{E}{2} & F \end{vmatrix}$$

  - If M is 0, then your conic is degenerate

5.7 Invariances

As you rotate a conic while maintaining its center, there are some properties that hold to the general form of the equations

- Before rotating, conic can be expressed as:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

- After rotating, conic can be expressed as:

$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0$$

Properties

1. $A + C = A' + C'$
2. $B^2 - 4AC = B'^2 - 4A'C'$
3. $D + E = D' + E'$ (assumes center to be at origin)
4. $F = F'$ (assumes center to be at origin)

6 Logs and Exponents

6.1 Review

Let's review the basic properties from algebra II:
Exponents:

$$x^a \cdot x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad x^0 = 1 \qquad x^{-a} = \frac{1}{x^a}$$

$$(x^a)^b = x^{ab} \qquad (xy)^a = x^a \cdot y^a \qquad \left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \qquad \left(\frac{x}{y}\right)^{-a} = \left(\frac{y}{x}\right)^a$$

Logarithms:

$$\log_x a + \log_x b = \log_x(ab) \qquad \log_x a - \log_x b = \log_x\left(\frac{a}{b}\right) \qquad a\log_x b = \log_x(b^a)$$

$$\log_x y = \frac{\log y}{\log x} \qquad \log_x y = \frac{1}{\log_y x} \qquad \log_{x^n} y = \frac{1}{n}\log_x y$$

Ex 1: Simplify $\log_4 9 \cdot \log_{343} 25 \cdot \log_3 7 \cdot \log_5 64$
Solution:

$$\begin{aligned}
&\log_4 9 \cdot \log_{343} 25 \cdot \log_3 7 \cdot \log_5 64 \\
=\ &\frac{2\log 3}{2\log 2} \cdot \frac{2\log 5}{3\log 7} \cdot \frac{\log 7}{\log 3} \cdot \frac{6\log 2}{\log 5} \\
=\ &\frac{2\log 3}{\log 3} \cdot \frac{2\log 5}{\log 5} \cdot \frac{\log 7}{3\log 7} \cdot \frac{6\log 2}{2\log 2} \\
=\ &4
\end{aligned}$$

Ex 2: Solve $9^x + 3^x - 6 = 0$.
Solution: This is a quadratic in disguise. We can rewrite as $(3^x)^2 + 3^x - 6 = (3^x + 3)(3^x - 2) = 0$. Since $3^x \neq -1$, we have $3^x = 2$, or $\boxed{x = \log_3 2}$

Ex 3: If $\log_4(x^6y^7) = 18$ and $\log_2(x^2y^{\frac{3}{2}}) = 12$, what is xy?

Solution: We rewrite $\log_4(x^6y^7)$ as $\frac{1}{2}\log_2(x^6y^7) = \log_2(x^3y^{\frac{7}{2}}) = 18$. We add this new equation to the second equation to get $\log_2(x^5y^5) = 30$. Using log properties, we get $5\log_2(xy) = 30$ or $\log_2(xy) = 6$. This means $\boxed{xy = 2^6 = 64}$.

6.2 Applications

Sometimes, questions will ask a logarithm/exponent question in the context of finance or population. You will need to memorize certain equations for various scenarios:

- When a population / bank account / any quantity grows **continuously**, use:

$$A = Pe^{rt}$$

  - A: amount after a time t
  - P: starting, or principle, amount
  - r: the rate at which the quantity grows
  - t: how long the quantity grows for
  - Note KEY WORD: CONTINUOUSLY

- When bank account / earnings / maybe a population has **compound** growth, use

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

  - A: amount after a time t
  - P: starting, or principle, amount
  - r: the rate at which the quantity grows
  - n: the number of times quantity is compounded per unit of time (usually years)
  - t: how long the quantity grows for
  - Key words would be semiannually, biannually, quarterly, etc...

Ex 1: 5 years ago, Greg put \$10,000 in a bank account that compounds continuously. Now, his account holds \$15,000. How long from now will it take for his current money to double?
Solution: Keyword *continuously*, so we use $A = Pe^{rt}$. We need to find the rate at which his money grows. Plugging our givens into the equation gets us $15,000 = 10,000e^{5r}$. Solving, we get $r = \frac{1}{5}\ln\frac{3}{2}$. We now solve the equation $30,000 = 15,000e^{rt}$. This gives us

$$t = \frac{1}{r}\ln 2 = \frac{5}{\ln\frac{3}{2}} \cdot \ln 2 = \frac{5(\ln 2)^2}{\ln 3}$$

Ex 2: Elisa is *also* putting her money in a bank, except her account will compound semi-annually at 10%. If she stores all her money in the account today, how long will it take in years from now for it to double?

Solution: Keyword *semiannually*, so we use $A = P\left(1+\frac{r}{n}\right)^{nt}$. We just simply plug in our givens and solve for t. Since we want the amount to double, A will be $2P$. So, $2P = P(1+\frac{0.1}{2})^{2t}$. This gives

$$2 = \left(\frac{21}{20}\right)^{2t} \qquad t = \frac{1}{2}\log_{\frac{21}{20}} 2 = \frac{\log 2}{2\log\frac{21}{20}} = \frac{\log 2}{2(\log 21 - \log 20)}$$

Notice how even though Greg's rate (abt %10) is lower than Elisa's rate, his money doubled a lot quicker because it is compounding continuously! (2.2 years vs Elisa's 7.1 yrs)

6.3 Euler's Number

The constant e can be expressed in a couple of different ways:

1. Exponents:
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$
2. Summation
$$e = \sum_{n=0}^{\infty}\frac{1}{n!} = \frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots$$

Oftentimes, you will have to manipulate the above expressions to get it in terms of e in order to simplify
Ex 1: Evaluate $\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^n$
Solution: We notice the given expression extremely similar to the definition of the exponent definition of e. We need to manipulate it to get it in the $\frac{1}{n}$ and raised to the n form:

$$\lim_{n\to\infty}\left(1-\frac{1}{2n}\right)^n = \left(1+\frac{1}{(-2n)}\right)^n = \left(\left(1+\frac{1}{(-2n)}\right)^{-2n}\right)^{-\frac{1}{2}} = e^{-\frac{1}{2}}$$

Essentially, we forced the inside expression to be 1 plus a fraction (step 2), and then we raised that expression to the denominator of the fraction (step 3) in order to get it in the form of e.
Ex 2: Evaluate $\sum_{n=0}^{\infty}\frac{n+1}{n!}$
Solution: Let's split this fraction into 2 different summations:

$$\sum_{n=0}^{\infty}\frac{n}{n!}+\sum_{n=0}^{\infty}\frac{1}{n!}$$

We can simplify the 1st summation to have an inside of $\frac{1}{(n-1)!}$. In doing so, our index shifts one value to begin at $n=1$. Additionally, the 2nd summation is just the definition of e. We have:

$$\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!}\right)+e$$

Expanding out the first several terms of this summation gives:

$$\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots+e$$

We notice that this is also the definition of e. So, we have $e+e=\boxed{2e}$.

7 Matrices

7.1 Matrices with Equations

- In a system of equations, instead of multiplying one equation by some constant and then going through the canceling-out-variables process, we can use matrices to solve.
- In order to use matrices, the systems of equations must have an equal number of equations as we have variables (because we will use square matrices)
- We will use a 3 by 3 system of equations to demonstrate:

$$\begin{aligned} 2x+4y+z&=-3\\ x+y\qquad&=-1\\ -x+3y+3z&=1 \end{aligned}$$

Cramer's Rule states

$$x=\frac{D_x}{D_c}\qquad y=\frac{D_y}{D_c}\qquad z=\frac{D_z}{D_c}$$

- D_c represents the matrices of coefficients. This is a 3 by 3 matrix with entries corresponding to the coefficients in our systems of equations

$$D_c=\begin{bmatrix}2&4&1\\1&1&0\\-1&3&3\end{bmatrix}$$

  - The column on the left represents the x column
  - Similarly, the column in the middle represents the y column
  - The column on the right represents the z column
  - Computing the determinant gives $D_c=-2$

- We will now solve for D_x. To construct this matrix, copy down the entries of D_c, remove the 3 entries of the x column, and replace it for the 3 constants in the given system of equations. We get:

$$D_x=\begin{bmatrix}-3&4&1\\-1&1&0\\1&3&3\end{bmatrix}=-1$$

- Since $x = \frac{D_x}{D_c}$, we get $x = \frac{-1}{-2} = \frac{1}{2}$
- Try repeating this process for yourself by solving for y and z!

We can also use Cramer's rule to determine if a system has 0, 1, or infinitely many solutions

- Pick a variable to solve for. If you choose x, compute $x = \frac{D_x}{D_c}$
  - If D_x and D_c do not equal 0, there is 1 solution
  - If $D_x \neq 0$ but $D_c = 0$, there is no solution
  - If $D_x = D_c = 0$, there are infinitely many solutions

7.2 Inverse Matrices

In Mu Alpha Theta, test writers only ask you to invert 2×2 and 3×3 matrices. If the dimension is higher, just skip the question...

- If the matrix is singular (determinant is 0), it is un-invertible
- If the matrix is not a square matrix, it is un-invertible

2 by 2 Matrices: We will use an arbitrary 2×2 matrix M to demonstrate. The steps will always be the same no matter what the entries are.

$$\text{Let } M = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$$

1. Calculate the determinant of M

$$2(-1) - (4)(1) = -6$$

2. Switch the entries of the main diagonal

$$\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix}$$

3. Negate the other 2 entries

$$\begin{bmatrix} -1 & 4 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$$

4. Multiply the resulting matrix by the reciprocal of the determinant of M

$$-\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$$

$\begin{bmatrix} \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & -\frac{1}{3} \end{bmatrix}$ is the inverse of $\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

3 by 3 Matrices Again, we will use an arbitrary 3×3 matrix M to demonstrate. The steps will always be the same no matter what the entries are.

$$\text{Let } M = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \\ 0 & 6 & 2 \end{bmatrix}$$

1. Find the determinant of M
   Using matrices of minors method:
   $$1(3 \cdot 2 - 6 \cdot -1) - 0(2 \cdot 2 - 0 \cdot -1) - 2(2 \cdot 6 - 0 \cdot 3) = -12$$

2. Construct a new matrix where the entry in row i column j is the minor of the original entry in row i column j
   $$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \\ 0 & 6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & 4 & 12 \\ 12 & 2 & 6 \\ 6 & 3 & 3 \end{bmatrix}$$

3. Let an entry in the new matrix be in row i column j. If $i + j$ is odd, multiply that entry by -1
   $$\begin{bmatrix} 12 & 4 & 12 \\ 12 & 2 & 6 \\ 6 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & -4 & 12 \\ -12 & 2 & -6 \\ 6 & -3 & 3 \end{bmatrix}$$

4. Transpose the matrix
   $$\begin{bmatrix} 12 & -4 & 12 \\ -12 & 2 & -6 \\ 6 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 12 & -12 & 6 \\ -4 & 2 & -3 \\ 12 & -6 & 3 \end{bmatrix}$$

5. Multiply by the reciprocal of the determinant of matrix M
   $$-\frac{1}{12} \begin{bmatrix} 12 & -12 & 6 \\ -4 & 2 & -3 \\ 12 & -6 & 3 \end{bmatrix} =$$
   $$\begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{4} \\ -1 & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

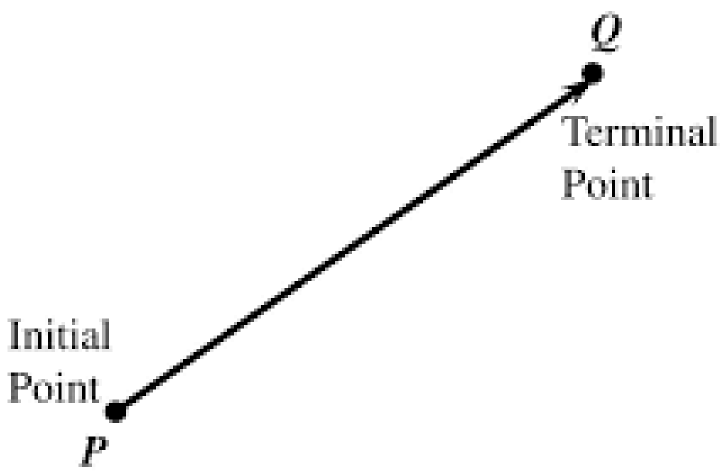
   The above matrix is the inverse of matrix M!

8 Vectors

8.1 Intro

All vectors, or directed line segment, have a direction and magnitude. They all have a starting point (initial) and an ending point (terminal).

- When a vector is in standard position, the initial point is located at the *origin*
- Usually, when a test question just drops a vector like $< 4, 3 >$, assume it to be in standard position unless stated otherwise
  - To visualize $< 4, 3 >$, plot the point $(4, 3)$ in a graph and draw an arrow starting at the origin to the point



Directed line segment

We can calculate a vector $\vec{v}$ from point $P(x_1, y_1)$ to $Q(x_2, y_2)$ as:

$$\vec{v} =< x_2 - x_1, y_2 - y_1 >$$

To calculate the magnitude of a vector $\vec{v} =< x, y >$, take the pythagorean distance:

$$|\vec{v}| = \sqrt{x^2 + y^2}$$

We have been showing examples in 2D space, but vectors can be in 3D as well!

- Test questions will usually be 2D or 3D, but vectors can be in whatever dimension you want
- We can add and subtract vectors! For example:

$$< 3, 4 > + < 7, -1 >=< 10, 3 >$$

$$< 10, 5 > - < 5, 10 >=< 5, -5 >$$

- The resulting vector after adding / subtracting does NOT necessarily have a magnitude equal to the sum of the 2 individual addends
- The magnitude of the resulting vector depends on the angle between the 2 vectors

- We can multiply vectors by *scalars.*
  - Multiplying a vector by a scalar constant k changes the magnitude of the vector by a factor of k.
  - Multiplying a vector by the scalar constant -1 reverses the direction of the vector

$$10 < \frac{5}{2}, -1 > = < 25, -10 >$$

Ex 1: Find the vector with terminal point $(-2, -3)$ and initial point $(1, 6)$
Solution: $\vec{v} = < -2 - 1, -3 - 6 > = \boxed{< -3, -9 >}$

Ex 2: What is the magnitude of $\vec{v} = < 2, 3, 6 >$?
Solution: The magnitude of 3D space vectors works the same as it does in 2D. $\sqrt{2^2 + 3^2 + 6^2} = \boxed{7}$

Ex 3: A cube with its faces aligned with the coordinate axes has its center at the origin. If it has a side length of 2, what is a vector representing its space diagonal?
Solution: There are two space diagonals. One diagonal has endpoints $(-1, -1, -1)$ and $(1, 1, 1)$. A vector through these points could be $\boxed{< -2, -2, -2 >}$. Note that this question is NOT a question they would ask in competition because the axes were not defined and there are many different orientations of the diagonal vector. As an exercise, try to find other vectors that represent the space diagonal.

8.2 Products

We stated that vectors can be added and subtracted, but ignored the product of vectors... This is because instead of the multiply and divide function between vectors, we have the dot product and cross product functions!

Dot Product

- The dot product can sort of be understood as a measure of how aligned two vectors are with respect to each other
- The dot product between two 2D vectors is:

$$< a, b > \cdot < c, d > = ac + bd$$

- It's very simple! The dot product between two 3D vectors is:

$$< a, b, c > \cdot < x, y, z > = ax + by + cz$$

- Notice how the dot product between two vectors results in a **scalar**.

- If the dot product between 2 vectors is 0, the vectors are **perpendicular**
- If the dot product between 2 vectors is large, it probably means they are closely aligned (sus..)
- You can only take the dot product between *2 vectors* of the *same dimension*
- We can take the dot product of higher dimension vectors and it will follow the same pattern as above

Recall how the dot product is a way to measure how closely aligned two vectors are.

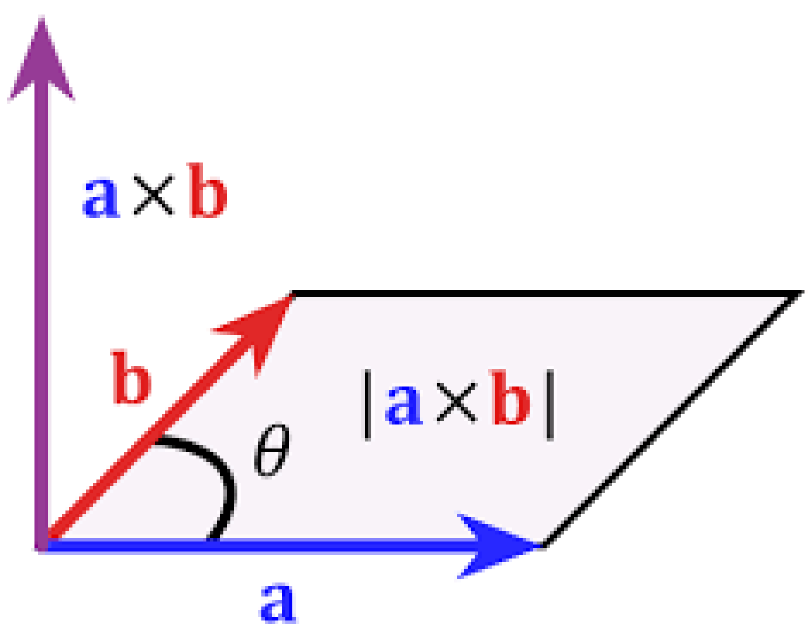
- In other words, it is (kind of) a measure of the angle θ between them
- The relationship between the dot product of $\vec{v}$ and $\vec{u}$ is

$$\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}| \cos \theta$$

- This is a SUPER important formula $\uparrow$

Cross Product

- The cross product of two vectors results in another vector that is perpendicular to both of them
- This involves vectors in 3D space



- The cross product between two vectors is:

$$< a, b, c > \times < x, y, z > = < \begin{vmatrix} b & c \\ y & z \end{vmatrix}, -\begin{vmatrix} a & c \\ x & z \end{vmatrix}, \begin{vmatrix} a & b \\ x & y \end{vmatrix} >$$

  - Notice how each entry of the cross product vector is a minor of the original 2 vectors

- If $\vec{w}$ is the cross product between $\vec{u}$ and $\vec{v}$, then

$$|\vec{w}| = |\vec{u}||\vec{v}| \sin\theta$$

- This is an important formula $\quad\uparrow$

- A common question writers like to put on tests is to find the equation of the plane going through 3 points
  - It is important to note that for a plane with the equation $Ax + By + Cz = D$, the vector that is normal (perpendicular) to the plane is $< A, B, C >$
  - The strategy for this question is to use the 3 points to create two vectors, use cross product to find the normal vector to the plane, and plug in A, B, C as well as one of the three points into the standard equation of the plane to solve for D

Ex 1: Evaluate $< 6, -1, 3 > \cdot < -2, -6, 2 >$
Solution: This is just $6(-2) + (-1)(-6) + 3(2) = \boxed{0}$

Ex 2: Find the sine of the acute angle formed by the intersection of the lines $y = 2x + 1$ and $2y = x - 10$
Solution: We can solve this question using vectors. This is the same thing as finding the acute angle between $< 1, 2 >$ and $< 2, 1 >$ (think about why!). Using $\vec{v} \cdot \vec{u} = |\vec{v}||\vec{u}| \cos\theta$, we have

$$1(2) + 2(1) = \sqrt{5}\sqrt{5}\cos\theta, \qquad \cos\theta = \frac{4}{5}$$

Using $\sin^2\theta + \cos^2\theta = 1$ and the fact that θ is acute, we get $\sin\theta = \boxed{\frac{3}{5}}$

Ex 3: A plane contains the points $A(1, 2, 3)$, $B(4, 0, -2)$, and $C(0, 1, 1)$. What is the equation of the plane?
Solution: We first create two vectors. I will use A as the initial point for both vectors. We get $< 3, -2, -5 >$ and $< -1, -1, -2 >$. We now find the vector that is perpendicular to the plane by calculating the cross product between said 2 vectors. We get $< -1, 11, -5 >$. Since this vector is normal to the plane, we know the plane is in the form $-x + 11y - 5z = D$. We know the point $(0, 1, 1)$ lies in the plane. Plugging this into our equation gives us $D = 6$. So, the plane is described by $\boxed{x - 11y + 5z = -6}$

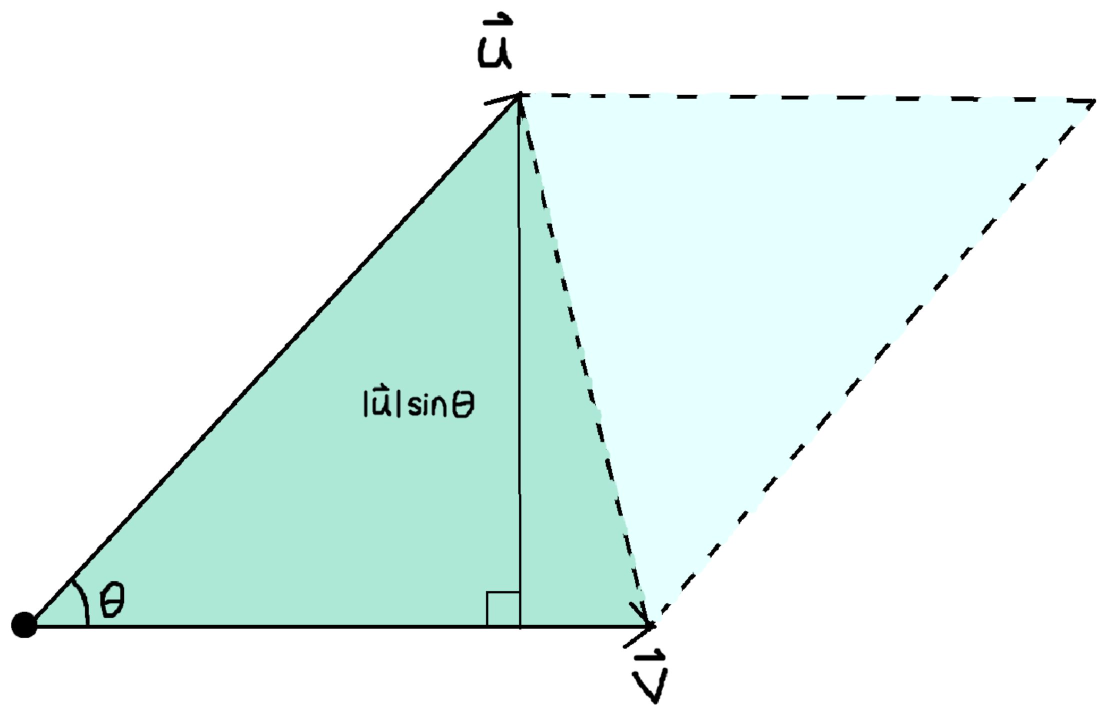
Ex 4: Consider vector $\vec{v} = < -4, 2, 4 >$. What is the vector that points in the opposite

direction of $\vec{v}$ that also has a magnitude of 30?
Solution: We know that by multiplying a vector by the scalar -1 reverses direction. This gives us $< 4, -2, -4 >$. The current magnitude of this vector is $\sqrt{4^2 + 2^2 + 4^2} = 6$. We can multiply this vector by the scalar $\frac{30}{6} = 5$ to get $\boxed{< 20, -10, -20 >}$

8.3 Area & Volume

Oftentimes, test writers will give you some vectors that define a figure and you'll be asked to find the area/volume of that thing

Area:
Parallelograms and triangles can be defined by 2 vectors visualized below:



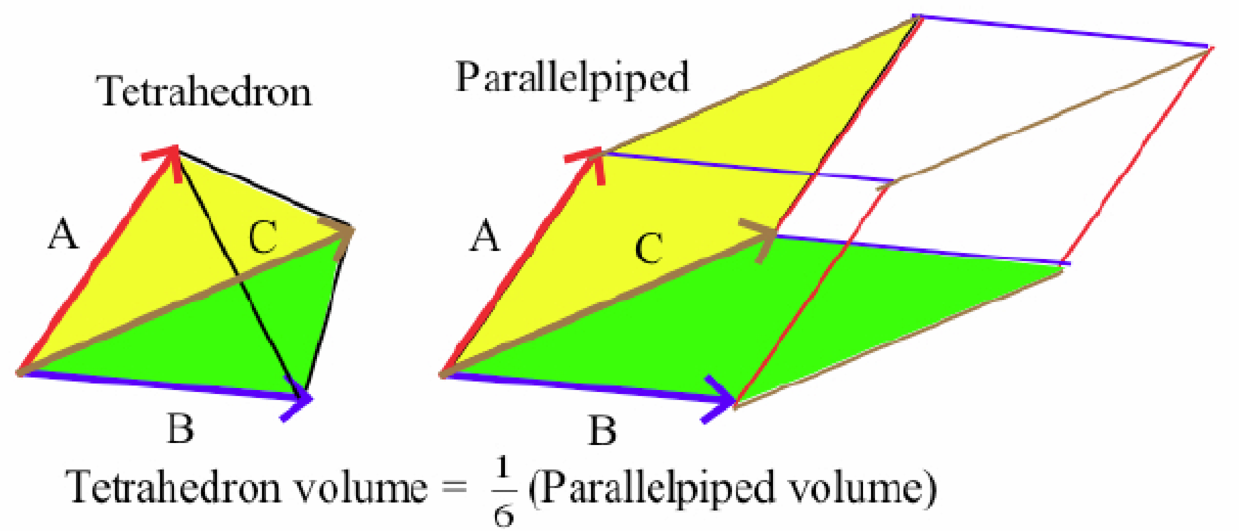
- The two vectors given to us are $\vec{u}$ and $\vec{v}$
- The darker shaded region is the **triangle** defined by these two vectors
  - Its area is calculated using one half base times height. Looking at the visual, the base has length $|\vec{v}|$ and height $|\vec{u}| \sin\theta$. This means the area is

$$\frac{1}{2}|\vec{u}||\vec{v}| \sin\theta$$

- Remember that $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin\theta$

- This means the area of the triangle defined by 2 vectors is one half times the magnitude of the cross product!
  - The entire region is the **parallelogram** defined by these two vectors.
  - The area of a parallelogram is just base times height, or $|\vec{u}||\vec{v}|\sin\theta$
  - This means the area of a parallelogram defined by 2 vectors is just the the magnitude of the cross product!

Volume:
Parallelepipeds and tetrahedrons are defined by three vectors as visualized below:



- If you are given 3 vectors $\vec{a}, \vec{b}, \vec{c}$ and asked to find the volume of the parallelepiped defined by them, simply calculate the scalar triple product (idk how to prove it, but just do it):

$$\vec{a} \cdot (\vec{b} \times \vec{c})$$

- Computing that might be kind of annoying, but this is the same thing as evaluating the determinant defined by these vectors (see in examples below)
- As the visual says, the tetrahedron defined by the vectors is 1/6th the volume of the parallelepiped
  - Just find 1/6th of the value of the determinant defined by the vectors
  - If the determinant is negative, just give the positive answer because volumes can't be negative
  - If the determinant is 0, it means your vectors are coplanar!

Ex 1: Find the area of the triangle defined by the points $A(4,3,2)$, $B(0,1,2)$, and $C(-2,0,0)$
Solution: Let's find 2 vectors that define this triangle. I will use point C as the common initial point. This gives us the vectors $< 6,3,2 >$ and $< 2,1,2 >$. Evaluating the cross product gives us $< 4,-8,0 >$. Half the magnitude of this is $\frac{1}{2}\sqrt{4^2+8^2} = \boxed{2\sqrt{5}}$

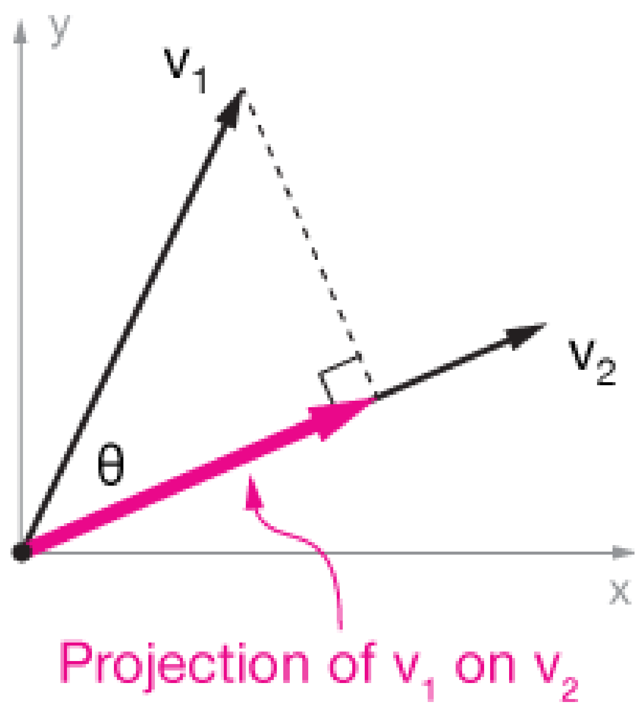
Ex 2: A tetrahedron is defined by the vectors $< 4,3,2 >$, $< 0,1,2 >$ and $< -2,0,0 >$. Find the volume.
Solution: We will just compute the determinant defined by these vectors. To set it up, just stack the vectors vertically on top of each other and fill up a 3×3 matrix with the corresponding entries. If we apply that here, we just evaluate the magnitude of:

$$\text{Volume} = \begin{vmatrix} 4 & 3 & 2 \\ 0 & 1 & 2 \\ -2 & 0 & 0 \end{vmatrix}$$

The magnitude of the determinant is $\boxed{8}$

8.4 Projections

Many times, test writers will ask to find the projection of a vector $\vec{v_1}$ onto a vector $\vec{v_2}$ (denoted as $\text{proj}_{\vec{v_2}}\vec{v_1}$) as shown below:



The pink vector is the projection of $\vec{v_1}$ onto $\vec{v_2}$

How to calculate it:

1. Calculate the magnitude of the projection

    - Remember that all vectors have a direction and magnitude. We are calculating the magnitude.
    - Using our right triangle trig skills, the magnitude, or length of the projection, is $|\vec{v_1}|\cos\theta$
        - It is basically finding the "horizontal" component, or leg of the right triangle in the image

2. Calculate the unit vector of the projection

    - We are now calculating the direction of our projection
    - Since we are projecting $\vec{v_1}$ *onto* $\vec{v_2}$, the projection will be in the same direction as $\vec{v_2}$.
    - The unit vector of $\vec{v_2}$ is simply $\vec{v_2}$ divided by its magnitude, or

$$\frac{1}{|\vec{v_2}|}(\vec{v_2})$$

3. Multiply them together
We are now combining the magnitude and direction to create the vector projection. This gives us

$$\frac{|\vec{v_1}|}{|\vec{v_2}|}\cos\theta(\vec{v_2})$$

4. Replace $\cos\theta$

    - A previous formula involving vectors and $\cos\theta$ was the dot product, which says $\vec{v_1}\cdot\vec{v_2} = |\vec{v_1}||\vec{v_2}|\cos\theta$. We can isolate $\cos\theta$ and get $\cos\theta = \frac{\vec{v_1}\cdot\vec{v_2}}{|\vec{v_1}||\vec{v_2}|}$. Plugging this into the expression from step 3 gets:

$$\left(\frac{\vec{v_1}\cdot\vec{v_2}}{|\vec{v_2}|^2}\right)\vec{v_2}$$

Instead of memorizing all of these steps, you can just use the formula to compute a vector projection. However, I recommend that you understand the steps for easy conceptualization so that you can just derive the formula on competition day.

Ex 1: Let $\vec{v} =< -3, 1, 5 >$ and let $\vec{u} =< 2, 4, 4 >$. Compute $\text{proj}_{\vec{u}}\vec{v}$.
Solution: Just substitute our values into the formula. Remember that the formula is

$$\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\right)\vec{u}$$

The dot product between $\vec{u}$ and $\vec{v}$ is $-3(2) + 1(4) + 5(4) = 18$. $|\vec{u}|^2 = 2^2 + 4^2 + 4^2 = 36$. $\vec{u} =< 2, 4, 4 >$. Plugging these values into the formula is

$$\frac{18}{36} < 2, 4, 4 >$$

Simplifying, we get $\boxed{< 1, 2, 2 >}$

Ex 2: Let $\vec{v} =< -3, 1, 5 >$ and let $\vec{u} =< 2, 4, 4 >$. Compute the component of $\vec{v}$ orthogonal to $\vec{u}$

Solution: Observe the right triangle in the picture of this section. The projection vector (pink vector) plus the orthogonal vector (dotted line) is equal to the vector that was projected ($\vec{v_1}$).

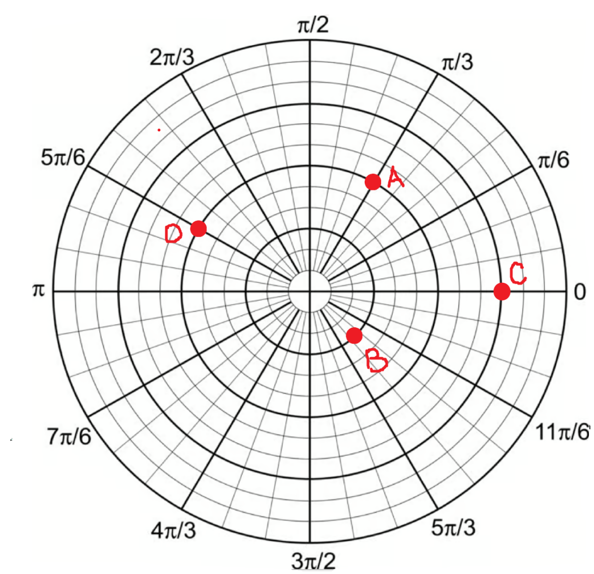
This question is asking us to find the vector that represents the dotted line, or $\vec{v} - \text{proj}_{\vec{u}}\vec{v}$. From example 1, we know $\text{proj}_{\vec{u}}\vec{v} =< 1, 2, 2 >$. So, our answer is $< -3, 1, 5 > - < 1, 2, 2 >=$ $\boxed{< -4, -1, 3 >}$

9 Polar

In the Cartesian plane, coordinates were plotted by going some units in the x direction and going some units in the y direction. In the Polar graph, coordinates are plotted by "facing" a direction that is an angle of θ away counterclockwise from the positive x-axis, and then "walking" a distance r in that direction.

9.1 Polar Coordinates

All polar coordinates are given in the form (r, θ)



To plot (r, θ), start the at the polar-axis (the "x-axis") and rotate θ radians counterclockwise

- If θ is negative, you will be rotating clockwise

After you have determined a direction, start at the pole (the "origin") and go r units in that direction.

- If r is negative, you will be going r units in the *opposite* direction that you previously determined

The following points are labeled in the diagram above:

1. $A(2, \frac{\pi}{3})$
2. $B(1, -\frac{5\pi}{4})$
3. $C(-3, \pi)$
4. $D(-2, -\frac{\pi}{6})$

Aside from graphing polar coordinates, we should also know how to convert from the polar system to rectangular (another way of saying Cartesian).

- If my polar coordinate is (r, θ), the coordinate in the Cartesian plane would be $(r\cos\theta, r\sin\theta)$
- You should know that $x = r\cos\theta$
- You should know that $y = r\sin\theta$
  - Super important $\uparrow$

9.2 Polar Equations

Just as there are equations in the Cartesian plane, equations can also be expressed in the polar plane.

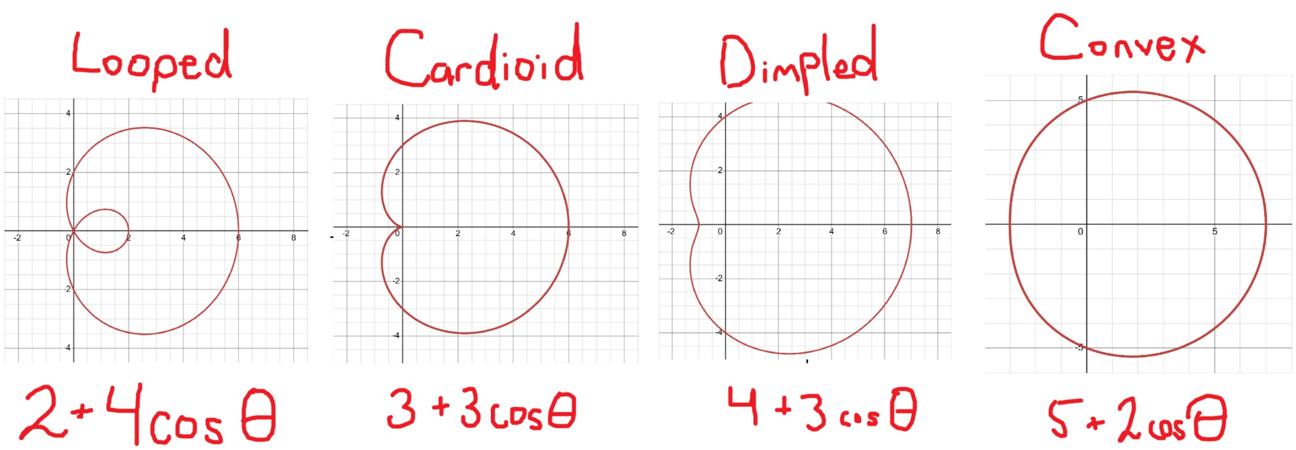
- Let's begin by finding the equation of a line. We will examine $3x + 2y = 1$
  - Recall that $x = r\cos\theta$ and $y = r\sin\theta$. Lets plug in these values of x and y into our line
  $$3r\cos\theta + 2r\sin\theta = 1$$
  - Isolating the r term:
  $$r = \frac{1}{3\cos\theta + 2\sin\theta}$$
- This process is the same for all rectangular graphs. Just plug in $x = r\cos\theta$ and $y = r\sin\theta$

9.3 Limaçons

- limaçons are a special type of polar graph that is always in the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$
  - If it is cos, then the limaçon is oriented horizontally
  - If it is sin, then the limaçon is oriented vertically

There are 4 types of limaçons that you will be asked to identify in competition:

1. **Looped** limaçon: $a < b$
2. **Cardioid**: $a = b$
3. **Dimpled** limaçon: $b < a < 2b$
4. **Convex** limaçon: $a \geq 2b$



How to graph limaçons:

1. Compare a and b and identify which of the four types it is
2. Plug in your four quadrental angles $(0, \frac{\pi}{2}, \pi, \frac{3\pi}{2})$ into the equation
3. Roughly sketch your graph going through the 4 points by tracing out the general shape of the identified type of limaçon

9.4 Other polar graphs

Circles:

- Circles can be represented in two ways in polar
  1. $r =$ radius

2. $r = a\cos(\theta + \phi)$ or $a\sin(\theta + \phi)$
    - The diameter of the circle in form 2 has a length of a

Rose curves:

Rose curves are in the following form (a must be an integer greater than 2):

$$r = a\sin(b\theta) \text{ or } a\cos(b\theta)$$

- As it's name implies, the graph of rose curves look like it has petals
- The number of petals the graph has depends on the parity of b
    - If b is odd, the rose curve will have b petals
    - If b is even, the rose curve will have $2b$ petals
- A typical question would be like "which of the following is not a possible number of petals a rose curve can have?"
    - Select the answer choice that is equivalent to $2 \mod 4$

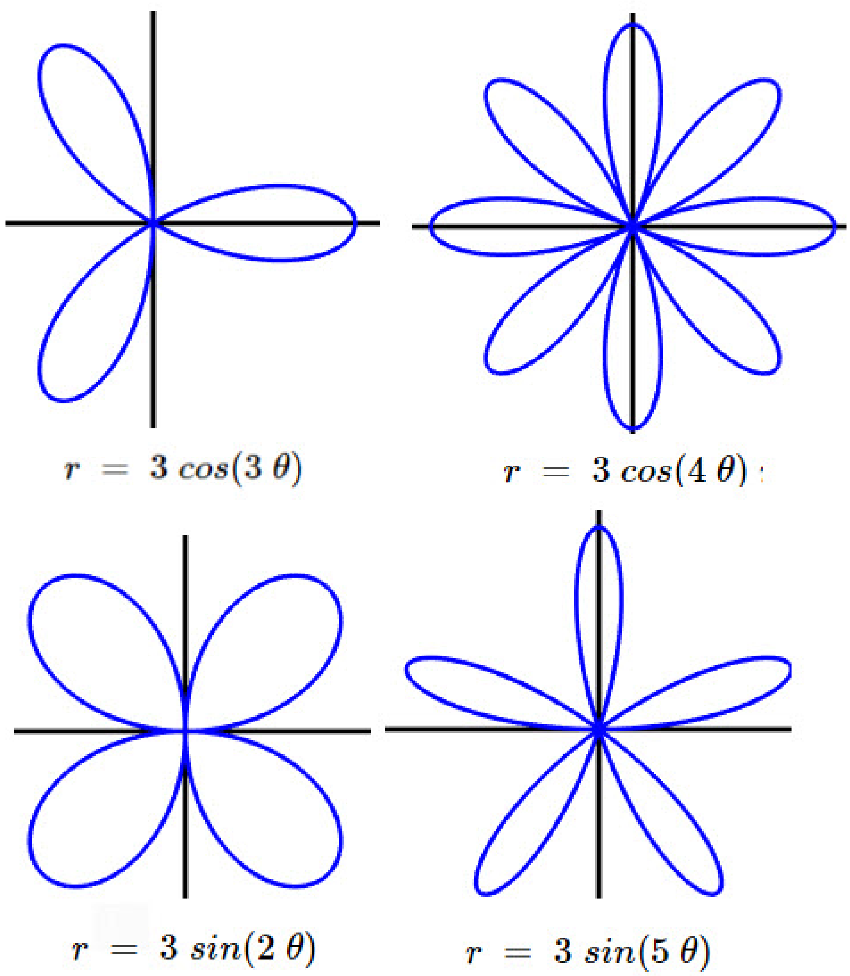


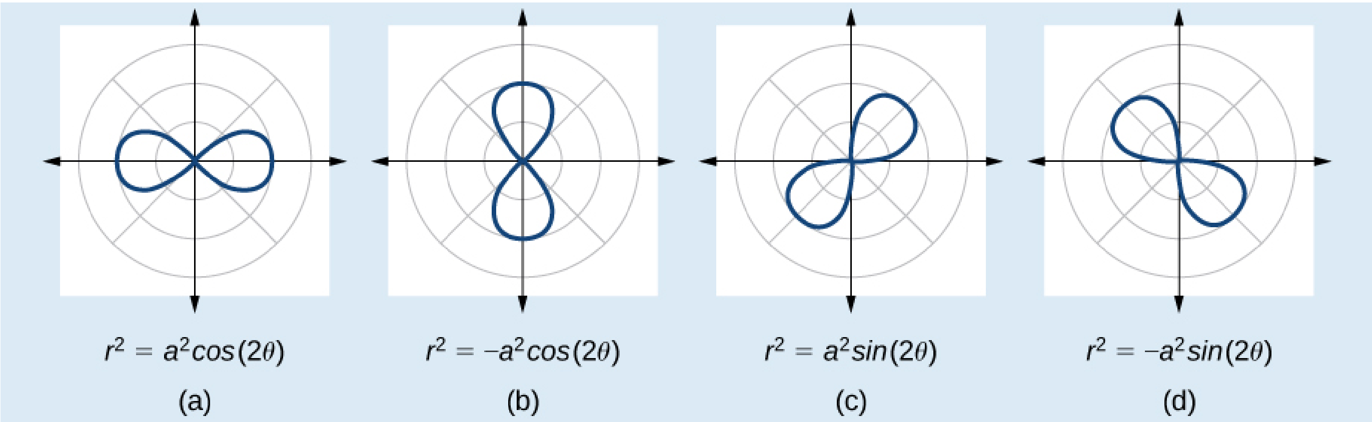
Figure 2.

Lemniscates:

- Lemniscates have the shape of an infinity
- They are represented as:

$$r^2 = a\cos(2\theta) \text{ or } a\sin(2\theta)$$

- 99% of the time, you will not have to worry about phase shifts for lemniscates

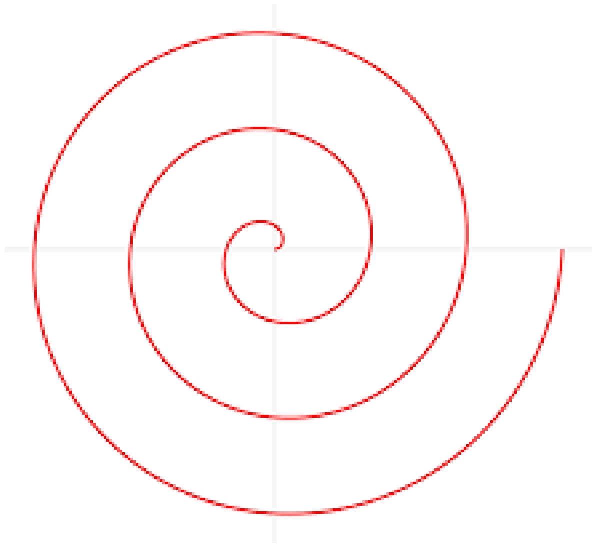


Spiral of Archimedes:

If θ ranges from $(-\infty, \infty)$, the spiral of archimedes can have either 2 arms or 1 arm.

- The one arm spiral is in the form $r = a|\theta|$
- The two arm spiral is in the form $r = a\theta$

The following image shows a one arm spiral. The two arm spiral would also include the reflection of this graph across the line $\theta = \frac{\pi}{2}$



9.5 Complex Numbers

- Usually, we deal with complex numbers in the form $a + bi$ where a and b are real numbers.
  - In Precalc, you will have to be able to convert from the $z = a + bi$ form to the polar form:
  $$z = r\text{cis}(\theta)$$
  - cisθ stands for $\cos\theta + i\sin\theta$
  - r is equal to the magnitude of z
  $$r = |z| = \sqrt{a^2 + b^2}$$
  - θ is equal to $\tan^{-1}\left(\frac{b}{a}\right)$
  - $\text{cis}(\theta + 2\pi k) = \text{cis}(\theta)$
    * Assume k is an integer
    * This is because sin and cosine have periods of 2π

Ex 1: Convert the polar coordinates $(6, \frac{\pi}{3})$ to rectangular form.
Solution: In rectangular, this is equal to $(6\cos\frac{\pi}{3}, 6\sin\frac{\pi}{3})$, which is $\boxed{(3, 3\sqrt{3})}$

Ex 2: Rewrite $4\text{cis}(-\frac{\pi}{6})$ in the form $a + bi$.
Solution: This is $4\cos(-\frac{\pi}{6}) + 4i\sin(-\frac{\pi}{6})$, which is $\boxed{2\sqrt{3} - 2i}$

Ex 3: Convert $-\sqrt{2} + i\sqrt{2}$ to polar form.
Solution: First we solve for r, which is the magnitude of the complex number. $r = \sqrt{2+2} = 2$. Next, we solve for θ, which is $\tan^{-1}\left(\frac{\sqrt{2}}{-\sqrt{2}}\right) = -\frac{\pi}{4}$. So, $-\sqrt{2} + i\sqrt{2} = \boxed{2\text{cis}\left(-\frac{\pi}{4}\right)}$

9.6 de Moivre's theorem

cisθ can alternatively be expressed as:

$$\text{cis}\theta = e^{i\theta}$$

Because of this alternate expression, we can derive several properties of the cis function. Consider the complex numbers:

$$z_1 = \text{cis}\theta_1 \qquad z_2 = \text{cis}\theta_2$$

Let's compute $z_1 z_2$:

$$z_1 = \text{cis}\theta_1 = e^{i\theta_1} \qquad z_2 = \text{cis}\theta_2 = e^{i\theta_2}$$

We have $z_1 z_2 = e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$
This is equivalent to $\text{cis}(\theta_1+\theta_2)$. So, we have shown a multiplicative property of complex numbers:

$$\text{cis}\theta_1 \cdot \text{cis}\theta_2 = \text{cis}(\theta_1+\theta_2)$$

By a similar logic, we can also show that:

$$\frac{\text{cis}\theta_1}{\text{cis}\theta_2} = \text{cis}(\theta_1-\theta_2)$$

Recall that $\text{cis}\theta_1 \cdot \text{cis}\theta_2 = \text{cis}(\theta_1+\theta_2)$. With this in mind, examine the following pattern:

- $r\text{cis}\theta \cdot r\text{cis}\theta = (r\text{cis}\theta)^2 = r^2\text{cis}(\theta+\theta) = r^2\text{cis}(2\theta)$
- $r\text{cis}\theta \cdot r\text{cis}\theta \cdot r\text{cis}\theta = r^3(\text{cis}\theta)^3 = r^3\text{cis}(\theta+\theta+\theta) = r^3\text{cis}(3\theta)$
- $r\text{cis}\theta \cdot r\text{cis}\theta \cdot r\text{cis}\theta \cdot r\text{cis}\theta = r^4(\text{cis}\theta)^4 = r^4\text{cis}(\theta+\theta+\theta+\theta) = r^4\text{cis}(4\theta)$
- And so on

We can arrive at an interesting conclusion from this. If we let $z = a+bi = r\text{cis}\theta$, then:

$$z^n = r^n\text{cis}(n\theta)$$

This is an especially powerful property with complex numbers. Instead of expanding out $(a+bi)^n$, we can convert to polar and easily compute r^n and $\text{cis}(n\theta)$.

Ex 1: Express $4e^{-\frac{\pi i}{4}}$ in the form $a+bi$.
Solution: We know $e^{i\theta} = \text{cis}(\theta)$. In this case, $\theta = -\frac{\pi}{4}$. So, we compute $4\text{cis}\left(-\frac{\pi}{4}\right) = 4(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i) = \boxed{2\sqrt{2} - 2i\sqrt{2}}$

Ex 2: Simplify $6\text{cis}\left(\frac{3\pi}{5}\right) \div 12\text{cis}\left(\frac{\pi}{10}\right) \cdot 4\text{cis}\left(\frac{7\pi}{12}\right) \div 2\text{cis}\left(-\frac{\pi}{4}\right)$
Solution: I recomend solving for the real component first. This is just $6 \div 12 \cdot 4 \div 2 = 4$. The complex component is equal to $\text{cis}\left(\frac{3\pi}{5} - \frac{\pi}{10} + \frac{7\pi}{12} - \frac{-\pi}{4}\right) = \text{cis}\left(\frac{4\pi}{3}\right)$. Multiplying these two together gives $4\text{cis}\frac{4\pi}{3} = \boxed{-2-2i\sqrt{3}}$

Ex 3: Simplify $(-\frac{1}{2} + \frac{\sqrt{3}}{2}i)^{2025}$
Solution: Whenever we see a complex number raised to a humongous power, always convert to cis. $-\frac{1}{2} + \frac{\sqrt{3}}{2}i = \text{cis}\left(\frac{2\pi}{3}\right)$. Raising this to the 2025 power is $\text{cis}\left(\frac{2\pi}{3} \cdot 2025\right) = \text{cis}\left(\frac{4050\pi}{3}\right) = \text{cis}(1350\pi) = \boxed{1}$

10 Sequences and Series

Please note that most of this content is from Alg II. Reference the Alg II guide for a more in depth analysis.

10.1 Binomials

Binomials can be written as $(a+b)^n$

- There are $n+1$ terms in the simplified binomial expansion
- The kth term with the first term being a^n is

$$a_k = (a)^{n-k+1}(b)^{k-1}\binom{n}{k-1}$$

Example: The 4th and 7th terms of the binomial $(a+b)^n$ have the same coefficient. What is the 3rd term?
Solution: If the 4th and the 7th terms have the same coefficient, this means the 3rd/8th, 2nd/9th, and 1st/10th terms, pairwise, have the same coefficients as well. Since there are 10 terms, this means $n = 9$. The 3rd term in this expansion is $a^7b^2\binom{9}{2} = \boxed{45a^7b^2}$

10.2 Arithmetic Sequences

An arithmetic sequence is a sequence of numbers such that the difference between consecutive terms is a constant.
The following is an arithmetic sequence:

$$a, a+d, a+2d, a+3d, a+4d, \ldots$$

It has a common difference of d

- We denote the first term as a
- We denote the common difference as d
- The nth term is $a_n = a + d(n-1)$

If we are asked to find the sum of the first n terms of an arithmetic sequence, we can make the following observation:

- Write down any arithmetic sequence
- Pair the 1st and last term, the 2nd and 2nd to last term, the 3rd and 3rd to last term, and so on
- Sum the numbers within each pair. Notice how each pair has the same sum
- The sum of each pair is equal to $a + a + d(n-1) = 2a + d(n-1)$
- The number of pairs is equal to $\frac{n}{2}$
- The sum of the first n terms is therefore equal to

$$[2a + d(n-1)] \cdot \frac{n}{2}$$

Another way of thinking about it: The sum of the first n terms of an arithmetic series is the average of the first and last terms times n:

$$S_n = \frac{a + a_n}{2}(n)$$

Ex 1: The first 4 terms of an arithmetic sequence is: $-5, -1, 3, 7$. What is the 20th term?
Solution: The first term is -5 and the common difference is $d = 4$. The 20th term is: $-5 + 19(4) = \boxed{71}$

Ex 2: The nth term of an arithmetic sequence is $5 + 4n$. What is the sum of the first 20 terms?
Solution: We find the average of the first and last terms and multiply by 20. $a = 9$ and $a_n = 85$. The average is 47. Multiplying by 20 gets us $\boxed{940}$

10.3 Geometric Sequences

A geometric sequence is a sequence of numbers such that the quotient between consecutive terms is a constant.
The following is a geometric sequence:

$$a, ar, ar^2, ar^3, ar^4, \ldots$$

This sequence as a common ratio of r

- We denote the first term as a
- We denote the common ratio as r
- The nth term is $a_n = ar^{n-1}$

For a geometric sequence, we can be asked to find the sum of a finite number of terms or an infinite number of terms.
Infinite:

- The common ratio of the sequence MUST satisfy $|r| < 1$
- The sum of an infinite sequence has the formula:

$$S = \frac{a}{1 - r}$$

Example: Compute $81 + 27 + 9 + 3 + 1 + \frac{1}{3} + \ldots$
Solution: The first term is 81 and the common ratio is $\frac{1}{3}$. Using our formula, we get $\boxed{\frac{243}{2}}$

Finite:

- The sum of a finite geometric series is essentially the difference between two infinite geometric series
- Again, the common ratio of the sequence MUST satisfy $|r| < 1$

Example: Let the first term of a geometric series be 2 and the common ratio be 2. Find the sum of the first 11 terms.
Solution: The series we seek to evaluate is

$$2 + 4 + 8 + 16 + \cdots + 2048$$

At first glance, we might have to evaluate this by hand because we can't use the formulas since our common ratio of 2 is greater than 1. However, if we rearrange the series to be:

$$2048 + 1024 + \ldots 8 + 4 + 2$$

Suddenly, the common ratio is $\frac{1}{2}$. If we think about this as the difference between two infinite series, we can find the sum of

$$2048 + 1024 + 512 + \ldots$$

and subtract it from

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots$$

Using our formulas, we get the first series evaluates to 4096 and the second series evaluates to 2. The difference is $\boxed{4094}$

- Reflecting on the method above, the sum of the first n terms of a geometric series with first term a and common ratio $|r| < 1$ is

$$\frac{a}{1-r} - \frac{ar^n}{1-r} = \frac{a - ar^n}{1-r}$$

10.4 Sequences & Series to infinity

Some test questions may ask if a sequence or series converges

- This is a way of asking "as the number of terms increases, does the sequence/series approach a constant number?"

The easiest way to determine the answer to this question is to derive the formula of the nth term (for a sequence) or the formula of the sum of the first n terms (for a series) and take the limit as n goes to infinity
General rule of thumb:

- If the common difference of an arithmetic series does not equal 0, then both the sequence and series **diverges**

- If the common ratio of a geometric series follows $|r| < 1$, then both the sequence and the series **converges**

Example: There exists a sequence such that the sum of the first n terms is $S_n = \frac{n^2}{n^2-1}$. Do the terms of the sequence converge?
Solution: We need to find the formula of the nth term. The trick to this question is subtracting the sum of the first $n-1$ terms from the sum of the first n terms. This is because

$$S_n - S_{n-1} = a_n + a_{n-1} + a_{n-2} + a_{n-3} + \cdots - (a_{n-1} + a_{n-2} + a_{n-3} + \ldots) = a_n$$

Computing $S_n - S_{n-1}$:

$$\frac{n^2}{n^2-1} - \frac{(n-1)^2}{(n-1)^2-1} = \frac{1-2n}{(n^2-1)(n^2-2n)}$$

Because the higher degree term is in the denominator, the sequence does indeed converge. So, $\boxed{\text{yes}}$, it converges to 0.

10.5 Induction

Induction is a very rare topic that is tested in the Alpha division. For this reason, this section contains surface level information.
Steps of induction:

1. **Base case:** Show that $P(1)$ is true
2. **Hypothesis:** Assume that $P(n)$ is true
3. **Induct:** Show that if $P(n)$ is true, then $P(n+1)$ is true

Example: Use induction to prove that the sum of the first n integers is $\frac{n(n+1)}{2}$
Solution:

1. We observe that $1 = \frac{1(1+1)}{2} = 1$
2. We assume that the sum of the first n integers is $\frac{n(n+1)}{2}$
3. We now calculate $1 + 2 + 3 + \ldots n + (n+1)$. From our hypothesis, we can substitute in the sum of the first n integers to get:

$$\frac{n(n+1)}{2} + (n+1)$$

Manipulating the above expression gets us

$$\frac{(n+1)(n+2)}{2} = \frac{(n+1)[(n+1)+1]}{2}$$

Because our assumption holds for $n+1$, by induction, the sum of the first n integers is $\frac{n(n+1)}{2}$

11 Probability and Statistics

The only new probability related concept from Theta to Alpha is the z scores. For this reason, surface level probability is contained in this study guide.

11.1 Basic Prob

The probability of something occurring is the number of favorable ways divided by the total number of ways.

- The probability of an event is always a number greater than or equal to 0 and less than or equal to 1
- The **complement** of an event is $1-p$, where p is the probability of the event occurring
  - The complement of an event is the probability that the event does *not* occur

Example: Chiikawa has a fair 20 sided dice labeled with integers from 1 to 20. What is the probability Chiikawa rolls a prime number?
Solution: There are ways 8 to roll a prime number : 2, 3, 5, 7, 11, 13, 17, 19. The probability is $\frac{8}{20} = \boxed{\frac{2}{5}}$

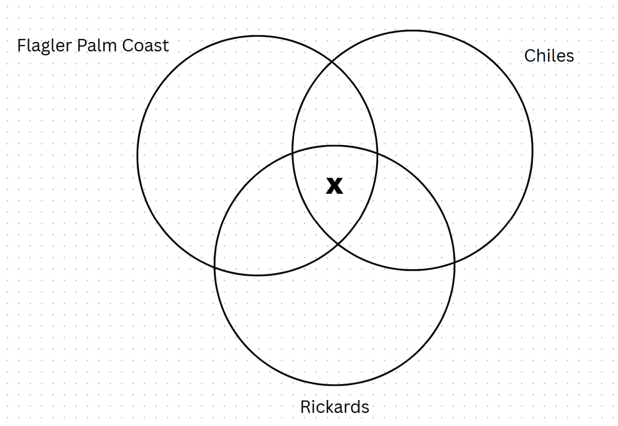
11.2 Venn Diagrams

Sometimes, questions will be loaded with so much information that is it too hard to keep track of every number. We can use Venn diagrams to help us visualize situations

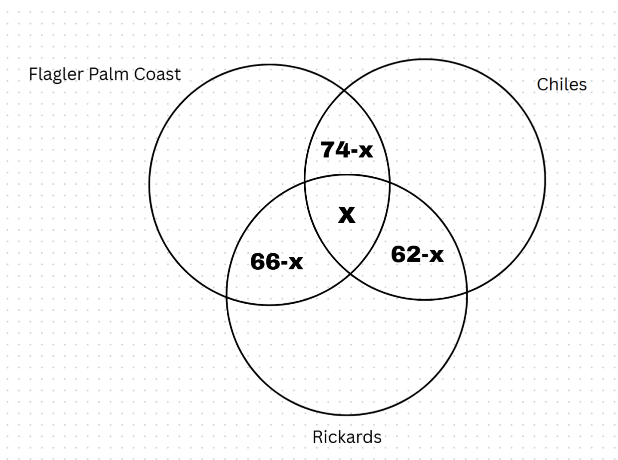
- You are expected to recognize the scenarios when a Venn Diagram is useful. You will usually use it in scenarios when an individual belongs in 0, 1, 2 or 3 "categories." Here is a list of common scenarios:
  - Students at a university can be enrolled in three different classes. Some students take 0 classes, some take only 1 of the three classes, some take 2 classes, and some take all 3
  - Students at a school participate in 3 different electives. A student either does 0, 1, 2, or all 3 electives
  - People in a group have up to 3 different hobbies. Some people do 1 hobby, some people do 2 different hobbies, and some people do all 3 hobbies.
- As you can tell, most scenarios involve *three* different categories

Example: A group of 100 students toured three different high schools. After all three tours, students were asked which of the high schools they enjoyed. 85 students said they enjoyed Flagler Palm Coast, 83 students enjoyed Chiles, and 74 enjoyed Rickards. 74 students enjoyed both Flagler and Chiles, 66 enjoyed both Flagler and Rickards, and 62 enjoyed both Chiles and Rickards. All 100 students reported they enjoyed at least one school. How many students enjoyed all 3 schools?

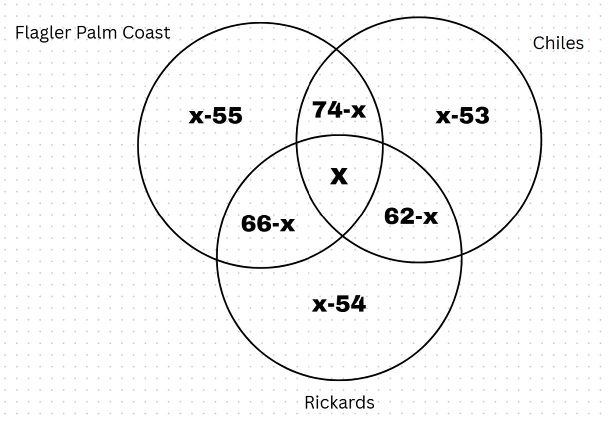
Solution: Since students can choose between 3 categories, we know to use a Venn Diagram to help us out. We can construct the visual below. The region where all 3 circles intersect represent how many students enjoyed all three schools. We can call this x.



Let's look at the football-looking region where 2 circles intersect. The values within each football should sum to 74, 66, and 62, each corresponding to the number of students who enjoyed 2 of the schools. So, the values in the empty region should be $74 - x$, $66 - x$, and $62 - x$ as follows:



We now look at each of the three circles individually. The sum of the values in each circle should add up to 85, 83, and 74. Doing some algebra, the values that go in the empty regions should be $x - 55, x - 53, x - 54$ as follows:



The sum of all the values in our Venn diagram should add up to 100 because there are 100 total students. So, we can set $x + 40 = 100$, which gives $\boxed{x = 60}$ students who enjoyed all three schools.

11.3 Mutually Exclusive Events

If two events A and B are mutually exclusive, then that means the probability of both events happening at the same time is 0. Algebraically, this can be written as:

$$P(A \cap B) = 0$$

The following are mutually exclusive events:

1. Flipping heads and flipping tails with one toss of a fair coin
2. Drawing a red card and drawing a spade from a standard 52 card deck
3. Sheldon passes a test and Sheldon fails a test

When solving for the probability of mutually exclusive events, it helps to draw a table:

	$P(A)$	$P(\bar{A})$	
$P(B)$	0	b	b
$P(\bar{B})$	a	c	$a + c$
	a	$b + c$	1

- b is the probability of event B occurring and event A not occurring (if B happens, then A can't happen if its mutually exclusive!)
- a is the probability of event A occurring and event B not occurring (if A happens, then B can't happen if its mutually exclusive!)
- c is the probability of neither event A nor B occurring

11.4 Independent Events

Two events are independent if the outcome of one event does NOT affect the outcome of another event.
In other words, if I have two independent events: event A with probability of a and event B with probability of b, the probability of event B occurring given that event A occurred is still b (sneakpeak of conditional probability)
The following are examples of independent events:

1. I roll an odd number on a standard dice and I flip heads on an unfair coin
2. I roll a 6 on one dice and I roll a 1 on another dice
3. Tomorrow's weather is rainy and you get a 100% on your math test
4. I become America's next president and I draw the ace of spades from a deck of cards

   Recall event A and B. The probability of event A occuring and event B occuring given they are independent is

$$P(A \cap B) = P(A) \cdot P(B) = ab$$

Example: Jeric has a red shirt and a white shirt. He also has red pants and green pants. Erry has a blue shirt and white shirt. He also has grey shorts, blue shorts, and red pants. If they independently select their outfit, what is the probability their fit is identical?
Solution: The only possible way for their clothes to be the same is if they both wear a white shirt and red pants. Since they select their outfit independently, we can just multiply the probabilities together:

$$\left(\frac{1}{2} \cdot \frac{1}{2}\right) \cdot \left(\frac{1}{2} \cdot \frac{1}{3}\right)$$

This evaluates to $\boxed{\frac{1}{24}}$

11.5 Conditional Probability

The conditional probability of an event occurring is the probability that the event occurs *given* other specified information. The extra information usually limits the total number of possible cases, which increases the probability of the event happening. Here are some scenarios demonstrating conditional probability:

1. I roll a 2 on a standard 6 sided dice given that the number rolled is even (total number of cases reduced from 6 to 3)
2. I draw the ace of spades from a 52 card deck given that the card is black (total number of cases reduced from 52 to 26)
3. I get above a 90% on the test given that I studied for the exam regularly (If I didn't study, the probability I get a 90 is much lower)

4. Monty Hall (google this!)

Example: Every day, Andrew, Billy, Caleb, Dean, and Eileen randomly sit in a row of 5 chairs before class starts. Whenever Eileen sits in the leftmost chair, her attention span increases. Yesterday, Caleb got in huge trouble. As a result, today, he must sit directly to the left of Dean. What is the probability that Eileen's attention span increases today during class?

Solution: On a normal day, there are 4! seating arrangements where Eileen sits in the leftmost chair among the 5! total cases. This gives a probability of $\frac{1}{5}$. However, we have the condition that Caleb must sit directly to the left of Dean, which limits the total number of cases. If we treat CD as a super person, there are $4! = 24$ total ways for A, B, CD, and E to sit down. Among these 24 ways, the number of ways Eileen can sit on the leftmost chair is 6 (easy to list them out). So, the probability is $\boxed{\frac{1}{4}}$

11.6 Binomial Distributions

A binomial distribution models the probability of achieving a certain number of successes after a fixed number n independent trials and a constant probability of success p. Note that there are only 2 events: success or no success
Scenarios of binomial distributions:

1. An fair coin is flipped 10 times. What is probability of flipping exactly 8 heads?
2. A machine puts stamps in the corner of envelopes. 40% of the time, the machine has a stamp misalignment. If 8 envelopes are sent through the machine, what is the probability exactly 2 stamps are misaligned?
3. 5% of all people at a sports convention play volleyball. If 20 people are randomly chosen, what is the probability exactly 1 of them plays volleyball?

Example: On any given day in a remote village, there is a 60% chance of rain. What is the probability that in a 7 day week, it rains 4 times?
Solution: We need to figure out the total number of ways to pick which 4 days in a week will rain. There are $\binom{7}{4} = \frac{7\cdot6\cdot5\cdot4}{4\cdot3\cdot2\cdot1} = 35$ ways to choose the order. In any given arrangement of 4 rainy days and 3 non-rainy days, the probability of this occurring is $(0.6)^4(0.4)^3$. Since there are 35 arrangements, the total probability is $35(0.6)^4(0.4)^3 \approx \boxed{0.29}$

11.7 z-scores

In any given data set of numbers, you can calculate, the mean, median, mode, and standard deviation, among other statistics (the other statistics are not relevant to Alpha)

- The standard deviation of a data set is a measure of how much the values vary from each other

- The data set {-4, 1000, 5000000, -12400000} has a huge standard deviation because the numbers are so far apart from each other; there is lots of variation
- The data set {8, 10, 11, 15, 3 } has a much smaller standard deviation because the numbers are closer together; there is little variation within the numbers

- Every value in a data set has an associated z-score
  - The z-score of a value x is the number of standard deviations σ that x is away from the mean $\bar{x}$ of the data set

$$z = \frac{x - \bar{x}}{\sigma}$$

  - For example, if a data set with a mean of 8 has a standard deviation of 5, a value of 23 is $\frac{23-8}{5} = 3$ standard deviations above the mean. This means it has a z-score of 3.
  - If a data set with a mean of -4 has a standard deviation of 3, a value of -10 is $\frac{-10-(-4)}{3} = -2$ standard deviations above the mean, or 2 standard deviations below the mean. This means it has a z-score of -2

Example: A hallway menace generated a random data set. He observed that 8 has a z score of 4 and 0.5 has a z score of -1. What is the standard deviation of his data set?
Solution: We can set up a system of equations:

$$4 = \frac{8 - \bar{x}}{\sigma}$$

$$-1 = \frac{0.5 - \bar{x}}{\sigma}$$

Solving this system gives us $\bar{x} = 2$ and $\boxed{\sigma = 1.5}$