FAMAT Algebra 2 Study Guide

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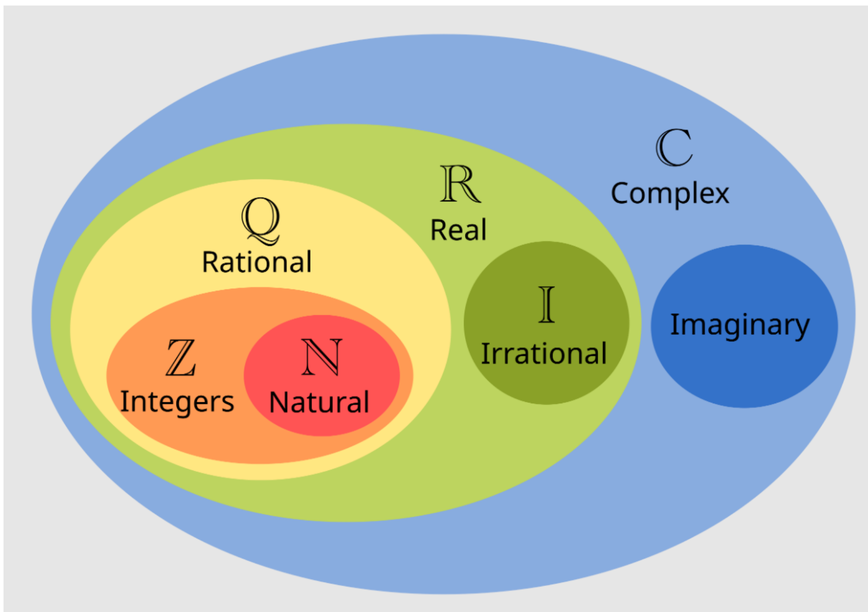
Contents

1 Real Numbers and Properties

1.1 Number Classifications

- **Natural**: $\mathbb{N} = \{1, 2, 3, \ldots\}$
- **Whole**: $\{0, 1, 2, 3, \ldots\}$
- **Integers**: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- **Rational**: $\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\right\}$
- **Irrational**: $\mathbb{R} \setminus \mathbb{Q}$ (e.g., $\pi, e, \sqrt{2}$)



1.2 Order of Operations (PEMDAS)

1. **P**arentheses
2. **E**xponents
3. **M**ultiplication & **D**ivision (left to right)
4. **A**ddition & **S**ubtraction (left to right)

Example: $5 + 3^2 \times (8 - 6)/2 = 5 + 9 \times 2/2 = 5 + 18/2 = 5 + 9 = 14$

2 Equations and Inequalities

2.1 Literal Equations

Solve for variable:
$V = \pi r^2 h \quad \Rightarrow \quad h = \frac{V}{\pi r^2}$
$P = 2l + 2w \quad \Rightarrow \quad w = \frac{P-2l}{2}$

2.2 Compound Inequalities

- **Conjunction (AND)**: $a < x < b$
  Example: $-3 \leq 2x + 1 < 5$
  $-4 \leq 2x < 4$
  $-2 \leq x < 2$
- **Disjunction (OR)**: $x < a$ or $x > b$
  Example: $3x + 2 < -4$ or $2x - 1 > 7$
  $3x < -6$ or $2x > 8$
  $x < -2$ or $x > 4$

2.3 Absolute Value Equations/Inequalities

Equations: $|3x - 2| = 7$
Case 1: $3x - 2 = 7 \Rightarrow x = 3$
Case 2: $3x - 2 = -7 \Rightarrow x = -\frac{5}{3}$
Inequalities: $|2x + 1| \leq 5$
$-5 \leq 2x + 1 \leq 5$
$-6 \leq 2x \leq 4$
$-3 \leq x \leq 2$

3 Linear Equations and Inequalities

3.1 Linear Functions

$$\begin{aligned} \text{Slope:} \quad & m = \frac{y_2 - y_1}{x_2 - x_1} \\ \text{Standard Form:} \quad & Ax + By = C \\ \text{Slope-Intercept Form:} \quad & y = mx + b \\ \text{Point-Slope Form:} \quad & y - y_1 = m(x - x_1) \end{aligned}$$

3.2 Parallel and Perpendicular Lines

$$\begin{aligned} \text{Parallel Lines:} \quad & m_1 = m_2 \\ \text{Perpendicular Lines:} \quad & m_1 \cdot m_2 = -1 \end{aligned}$$

Example: Line perpendicular to $y = -\frac{2}{3}x + 4$ through $(1, 5)$ has slope $m = \frac{3}{2}$. Equation: $y - 5 = \frac{3}{2}(x - 1)$.

3.3 Inverse Functions

- To find the inverse of $f(x)$:
    1. Replace $f(x)$ with y
    2. Swap x and y
    3. Solve for y
- Graphically: An inverse is the reflection across the line $y = x$
- The domain of f = range of f^{-1}, and vice versa

Example: Find inverse of $f(x) = 2x - 3$:

$$\begin{aligned} y &= 2x - 3 \\ x &= 2y - 3 \\ y &= \frac{x+3}{2} \\ \Rightarrow f^{-1}(x) &= \frac{x+3}{2} \end{aligned}$$

3.4 Parallel and Perpendicular Lines

$$\begin{aligned} \text{Parallel Lines:} \quad & m_1 = m_2 \\ \text{Perpendicular Lines:} \quad & m_1 \cdot m_2 = -1 \end{aligned}$$

Example: Find the line perpendicular to $y = -\frac{2}{3}x + 4$ that passes through $(1, 5)$.
New slope $m = \frac{3}{2}$. Using point-slope: $y - 5 = \frac{3}{2}(x - 1)$

3.5 Direct and Inverse Variation

$$\begin{aligned} \text{Direct Variation:} \quad & y = kx, \ k \neq 0 \\ \text{Inverse Variation:} \quad & y = \frac{k}{x}, \ k \neq 0 \end{aligned}$$

3.6 Functions

3.6.1 Set Notation and Intervals

A **set** is a collection of values. Example: $\{1, 2, 3, 4, 5\}$. Use $\in$ to indicate membership: $5 \in S$. Use $\notin$ to show a value is not in the set: $6 \notin S$.

Intervals: Use [ ] for included endpoints (closed) and ( ) for excluded endpoints (open). Examples:

- $0 < x \leq 5$ is written as $(0, 5]$
- $x \geq 0$ is $[0, \infty)$

Unions and Intersections: $\cup$ means union (OR), $\cap$ means intersection (AND).

Example:
Let $A = \{1, 2, 3, 4\}$, and $B = \{3, 4, 5, 6\}$

- $A \cup B = \{1, 2, 3, 4, 5, 6\}$ (All elements from either A or B)
- $A \cap B = \{3, 4\}$ (Only elements in both A and B)

3.6.2 Definition of a Function

A **function** gives one output for each input. Notation: $f(x)$.
Vertical Line Test: If a vertical line crosses a graph more than once, it is not a function.
Horizontal Line Test: If a horizontal line crosses a graph more than once, the function is not one-to-one and thus does not have an inverse that is a function.

3.6.3 Operations with Functions

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f \cdot g)(x) = f(x) \cdot g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ where } g(x) \neq 0$$

3.6.4 Domain and Range

Domain: All possible x-values (inputs)
Range: All possible y-values (outputs)

3.6.5 Composite Functions

Notation: $(f \circ g)(x) = f(g(x))$ — evaluate the inner function $g(x)$ first, then plug the result into $f(x)$.

Example: Let $f(x) = 2x + 3$ and $g(x) = x^2$. Then:

$$(f \circ g)(x) = f(g(x)) = f(x^2) = 2x^2 + 3$$

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 = 4x^2 + 12x + 9$$

3.6.6 Piecewise and Absolute Value Functions

Piecewise Example:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 3 \\ 6 - x & \text{if } x > 3 \end{cases}$$

Absolute Value Function:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

3.6.7 Continuity

A function is **continuous** if you can draw it without lifting your pencil.

3.7 Rational Functions

3.7.1 Definition and Asymptotes

A **rational function** is a ratio of polynomials: $f(x) = \frac{P(x)}{Q(x)}$

- **Vertical Asymptotes:** Set $Q(x) = 0$, but ensure $P(x) \neq 0$ at those values. **Example:** $f(x) = \frac{x+2}{x-3} \Rightarrow$ Vertical asymptote at $x = 3$
- **Holes (Removable Discontinuities):** Occur when $P(x)$ and $Q(x)$ share a common factor that cancels. **Example:** $f(x) = \frac{(x-1)(x+2)}{(x-1)(x-3)}$ has a hole at $x = 1$ (canceling factor), and a vertical asymptote at $x = 3$
- **Horizontal Asymptotes:** Determined by the degree of $P(x)$ and $Q(x)$:
  - $\deg(P) < \deg(Q)$: Horizontal asymptote at $y = 0$ **Example:** $f(x) = \frac{x}{x^2+1} \Rightarrow$ HA at $y = 0$
  - $\deg(P) = \deg(Q)$: Horizontal asymptote at $y = \frac{\text{leading coefficient of } P}{\text{leading coefficient of } Q}$ **Example:** $f(x) = \frac{3x^2+1}{2x^2-5} \Rightarrow$ HA at $y = \frac{3}{2}$
  - $\deg(P) > \deg(Q)$: No horizontal asymptote (consider slant instead)
- **Slant (Oblique) Asymptotes:** If $\deg(P) = \deg(Q)+1$, use polynomial long division. **Example:** $f(x) = \frac{x^2+1}{x-1}$. Long division yields $f(x) = x+1+\frac{2}{x-1}$, so the slant asymptote is $y = x + 1$

3.7.2 Domain of Composite Functions

To find the domain of $f(g(x))$:

1. Exclude x-values not allowed in $g(x)$
2. Then exclude any x such that $g(x)$ is not allowed in $f(x)$

Example: Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{x-2}$. Then $(f \circ g)(x) = f(g(x)) = \sqrt{\frac{1}{x-2}}$
Step 1: Domain of $g(x) = \frac{1}{x-2}$ excludes $x = 2$ (division by zero) **Step 2:** $f(g(x)) = \sqrt{\frac{1}{x-2}}$ requires $\frac{1}{x-2} \geq 0$
Solve: $\frac{1}{x-2} \geq 0 \Rightarrow x > 2$
Final Domain: $x > 2$

3.8 Even and Odd Functions

- **Even:** $f(-x) = f(x)$ (symmetric about the y-axis) **Example:** $f(x) = x^2 \Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$
- **Odd:** $f(-x) = -f(x)$ (symmetric about the origin) **Example:** $f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$
- **Neither:** If neither condition holds **Example:** $f(x) = x^2+x \Rightarrow f(-x) = x^2-x \neq f(x)$ and $\neq -f(x)$

Special Case: The zero function $f(x) = 0$ is the only function that is *both* even and odd.
How to Test: Plug in x and $-x$ into the function.

- If $f(-x) = f(x)$, the function is even.
- If $f(-x) = -f(x)$, the function is odd.
- If neither, then the function is neither even nor odd.

4 Systems of Equations and Inequalities

4.1 Systems of Equations

Systems of equations are sets of two or more equations with the same variables that you solve simultaneously.

Solution Methods:

- **Graphing:** Find points where the graphs of equations intersect.
- **Substitution:** Solve one equation for a variable and substitute into others.
- **Elimination:** Add or subtract equations to eliminate one variable.

Example (Elimination):

$$\begin{cases} 3x + 2y = 8 \\ 2x - y = 3 \end{cases}$$

Multiply the second equation by 2:

$$4x - 2y = 6$$

Add it to the first:

$$(3x + 2y) + (4x - 2y) = 8 + 6 \implies 7x = 14 \implies x = 2$$

Substitute back:

$$2(2) - y = 3 \implies y = 1$$

4.2 Systems of Inequalities

Systems of inequalities involve two or more inequalities with the same variables.

Solution: Find the region where all inequalities overlap, usually shaded on a graph.

Steps to Solve:

- Graph each inequality.
- Identify the overlapping shaded region.
- Determine boundary points if needed.

4.3 Determinants

2×2 Determinant For matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the determinant is

$$\det(A) = ad - bc.$$

3×3 Determinant **Rule of Sarrus:** For

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

compute

$$\det(A) = (aei + bfg + cdh) - (ceg + bdi + afh).$$

Cofactor Expansion: Expand along the first row:

$$\det(A) = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

General $n \times n$ Determinants Calculate determinants recursively by expanding minors and cofactors:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij},$$

where M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix after removing row i and column j.

4.4 Cramer's Rule

Cramer's Rule solves systems of linear equations using determinants when the coefficient matrix has a nonzero determinant.

For 2×2 Systems Given

$$\begin{cases} ax + by = e \\ cx + dy = f, \end{cases}$$

compute

$$\Delta = \text{determinant of matrix of coefficients} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

If $\Delta \neq 0$,

$$x = \frac{\det \begin{bmatrix} e & b \\ f & d \end{bmatrix}}{\Delta} = \frac{ed - bf}{\Delta}, \quad y = \frac{\det \begin{bmatrix} a & e \\ c & f \end{bmatrix}}{\Delta} = \frac{af - ec}{\Delta}.$$

For 3×3 Systems Given

$$\begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \\ a_3 x + b_3 y + c_3 z = d_3, \end{cases}$$

compute

$$D = \text{determinant of matrix of coefficients} = \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

If $D \neq 0$,

$$x = \frac{\det \begin{bmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{bmatrix}}{D}, \quad y = \frac{\det \begin{bmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{bmatrix}}{D}, \quad z = \frac{\det \begin{bmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{bmatrix}}{D}.$$

5 Radicals and Complex Numbers

5.1 Radical Operations

Radicals involve roots of numbers, such as square roots, cube roots, etc. Here are key points and examples:

- **Simplifying radicals:** Break down into prime factors and extract perfect powers.

$$\sqrt{50} = \sqrt{25 \times 2} = 5\sqrt{2}$$

$$\sqrt[3]{54} = \sqrt[3]{27 \times 2} = 3\sqrt[3]{2}$$

- **Division of radicals:** When dividing two radicals with the same index, combine under one radical.

$$\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$$

- **Powers of radicals:** Apply exponent to both the root and the radicand.

$$\left(\sqrt[4]{16}\right)^3 = \left(16^{\frac{1}{4}}\right)^3 = 16^{\frac{3}{4}} = (2^4)^{\frac{3}{4}} = 2^3 = 8$$

5.2 Rational and Irrational Numbers

- **Rational numbers** can be expressed as a fraction of two integers, e.g., $\frac{3}{4}$, 0.75.
- **Irrational numbers** cannot be expressed as a simple fraction and have non-repeating, non-terminating decimals, e.g., $\sqrt{2}$, π.

5.3 Complex Numbers

Complex numbers extend the real number system by including the imaginary unit i, defined as:

$$i = \sqrt{-1}, \quad i^2 = -1.$$

Powers of i:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = i^2 \cdot i = -i, \quad i^4 = (i^2)^2 = 1,$$

and the powers repeat every 4 steps:

$$i^5 = i, \quad i^6 = -1, \quad \text{etc.}$$

Operations with Complex Numbers: Given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$:

- **Addition/Subtraction:** Add or subtract real and imaginary parts separately.

$$(3 + 2i) + (1 - 4i) = (3 + 1) + (2 - 4)i = 4 - 2i$$

- **Multiplication:** Use distributive property and apply $i^2 = -1$.

$$(2 - i)(3 + 2i) = 2 \cdot 3 + 2 \cdot 2i - i \cdot 3 - i \cdot 2i = 6 + 4i - 3i - 2i^2 = 6 + i + 2 = 8 + i$$

- **Division:** Multiply numerator and denominator by the **complex conjugate** of the denominator to eliminate i from denominator.

  The **complex conjugate** of $a + bi$ is $a - bi$.

  Example:

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{(1+i)^2}{(1)^2 - (-i)^2} = \frac{1 + 2i + i^2}{1 - (-1)} = \frac{1 + 2i - 1}{2} = \frac{2i}{2} = i$$

The Argand Plane Complex numbers can be represented graphically on the **Argand plane**, where:

- The horizontal axis (x-axis) represents the **real part** a.
- The vertical axis (y-axis) represents the **imaginary part** b.

So, a complex number $a + bi$ corresponds to the point (a, b).

Magnitude and Argument

- **Magnitude** (or modulus) of $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}.$$

- **Argument** (or angle θ) is the angle made with the positive real axis:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right).$$

6 Polynomial Functions and Rational Expressions

6.1 Types and Naming of Polynomials

A **polynomial** is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the a_i are constants and n is a non-negative integer.

- **Degree:** The highest exponent n with nonzero coefficient.
- **Naming by degree:**
  - Degree 0: Constant polynomial, e.g. $f(x) = 5$.
  - Degree 1: Linear polynomial, e.g. $f(x) = 3x + 1$.
  - Degree 2: Quadratic, e.g. $f(x) = x^2 - 4x + 4$.
  - Degree 3: Cubic, e.g. $f(x) = x^3 - 3x^2 + 2$.
  - Degree 4: Quartic, e.g. $f(x) = x^4 + 2x^2 - 1$.
  - Degree 5: Quintic, e.g. $f(x) = x^5 - x + 3$.
- **Number of terms:**
  - Monomial: One term, e.g. $f(x) = 7x^3$.
  - Binomial: Two terms, e.g. $f(x) = x^2 - 9$.
  - Trinomial: Three terms, e.g. $f(x) = x^2 + 5x + 6$.
- **Monic Polynomial:** A polynomial is **monic** if the leading coefficient (the coefficient of the highest-degree term) is 1.
  Example: $f(x) = x^3 - 2x + 7$ is monic, but $f(x) = 3x^3 - x + 2$ is not.

6.2 Solving Quadratic and Higher-Order Polynomial Equations

Zeros and Multiplicity A **zero** (root) of a polynomial $f(x)$ is a value r such that $f(r) = 0$.

- **Multiplicity** of a zero refers to how many times that root appears as a factor:

$$(x - r)^m \text{ indicates root } r \text{ of multiplicity } m.$$

- If multiplicity m is odd, the graph crosses the x-axis at r. - If m is even, the graph touches but does not cross the axis at r.

Example: Let $f(x) = (x - 2)^2(x + 3)$.

- Factor form shows roots: $x = 2$ and $x = -3$.
- $x = 2$ has multiplicity 2 $\Rightarrow$ graph touches x-axis at $x = 2$.
- $x = -3$ has multiplicity 1 $\Rightarrow$ graph crosses x-axis at $x = -3$.

Discriminant (Quadratics) For a quadratic $ax^2 + bx + c = 0$, the **discriminant** is

$$\Delta = b^2 - 4ac.$$

Example: Solve $x^2 - 4x + 4 = 0$.

- Identify $a = 1$, $b = -4$, $c = 4$.
- Discriminant: $\Delta = (-4)^2 - 4(1)(4) = 16 - 16 = 0$.
- Since $\Delta = 0$, one real root with multiplicity 2.
- Use quadratic formula: $x = \frac{-(-4)\pm\sqrt{0}}{2(1)} = \frac{4}{2} = 2$.
- Root: $x = 2$.

6.3 Polynomial Functions and Their Graphs

- **Degree and leading coefficient affect end behavior:**
  - **Even degree:** both ends of the graph go in the same direction.
  - **Odd degree:** ends of the graph go in opposite directions.
  - **Positive leading coefficient:** right end of the graph rises.
  - **Negative leading coefficient:** right end of the graph falls.
- **Zeros and their multiplicities:**
  - A zero with **odd multiplicity** causes the graph to *cross* the x-axis at that point.
  - A zero with **even multiplicity** causes the graph to *touch* the x-axis and turn around.

6.4 Fractional Equations and Rational Functions

- A **rational function** is a ratio of two polynomials:

$$R(x) = \frac{P(x)}{Q(x)}.$$

Example: Let $R(x) = \frac{x^2-1}{x-1}$.

- Factor numerator: $x^2 - 1 = (x-1)(x+1)$.
- Cancel common factors: $\frac{(x-1)(x+1)}{x-1} = x + 1$, for $x \neq 1$.
- Domain excludes $x = 1$, since it causes division by zero.

6.5 Quadratic Inequalities and Their Graphs

To solve $ax^2 + bx + c > 0$:

- Find roots by solving $ax^2 + bx + c = 0$.
- Plot roots on a number line.
- Test values in each region.

Example: Solve $x^2 - 4 > 0$.

- Solve $x^2 - 4 = 0 \Rightarrow x = \pm 2$.
- Test interval $(-\infty, -2)$: pick $x = -3$, $(-3)^2 - 4 = 9 - 4 = 5 > 0$.
- Test $(-2, 2)$: pick $x = 0$, $0^2 - 4 = -4 < 0$.
- Test $(2, \infty)$: pick $x = 3$, $3^2 - 4 = 9 - 4 = 5 > 0$.
- Solution: $x < -2$ or $x > 2$.

6.6 Remainder and Factor Theorems

Remainder Theorem: Remainder of $f(x)$ divided by $x - c$ is $f(c)$.
Example: $f(x) = x^2 - 3x + 2$, divide by $x - 1$.

- Evaluate: $f(1) = 1^2 - 3(1) + 2 = 1 - 3 + 2 = 0$.
- Remainder $= 0$.

Factor Theorem: $x - c$ is a factor if $f(c) = 0$.

- Since $f(1) = 0$, $x - 1$ is a factor.

6.7 Fundamental Theorem of Algebra

Every polynomial of degree $n \geq 1$ has n roots in $\mathbb{C}$, counting multiplicity.
Example: $f(x) = x^2 + 1$

- Roots: $x = \pm i$ (since $x^2 = -1$).

6.8 Conjugate Root Theorem

If a polynomial has real coefficients and a complex root $a + bi$, it also has $a - bi$.
Example: $f(x) = x^2 - 2x + 2$

- Use quadratic formula: $x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$.
- Roots: $1 + i$ and $1 - i$.

6.9 Conjugate Radical Root Theorem

If a polynomial has rational coefficients and a root of the form $a + \sqrt{b}$, where b is not a perfect square, then it also has the conjugate root $a - \sqrt{b}$.

Example: $f(x) = x^2 - 4x + 1$

- Use quadratic formula: $x = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(1)}}{2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$.
- Roots: $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

6.10 Rational Root Theorem

If $f(x) = a_n x^n + \cdots + a_0$, rational roots $\frac{p}{q}$ satisfy:

- $p | a_0$, $q | a_n$.

Example: $f(x) = 2x^3 + 3x^2 - x - 6$.

- Constant term: -6. Possible p: $\pm 1, \pm 2, \pm 3, \pm 6$
- Leading coefficient: 2. Possible q: $\pm 1, \pm 2$
- Try $x = 1$: $2(1)^3 + 3(1)^2 - 1 - 6 = 2 + 3 - 1 - 6 = -2$
- Try $x = -1$: $2(-1)^3 + 3(-1)^2 - (-1) - 6 = -2 + 3 + 1 - 6 = -4$
- Try $x = 2$: $2(8) + 3(4) - 2 - 6 = 16 + 12 - 2 - 6 = 20$
- Try $x = 1$ again: mistake found earlier, recheck: $2(1)^3 + 3(1)^2 - 1 - 6 = 2 + 3 - 1 - 6 = -2$, still not a root.
- Try $x = 3$: $2(27) + 3(9) - 3 - 6 = 54 + 27 - 3 - 6 = 72$ too big.
- Try $x = -1$ again: not a root.
- Eventually try $x = 1$, mistake again; should try full synthetic division.

6.11 Descartes' Rule of Signs

- Number of sign changes in $f(x)$ = max number of positive real roots. - Number of sign changes in $f(-x)$ = max number of negative real roots.

Example: $f(x) = x^3 - 4x^2 + 5x - 2$

- Signs: $+ - + - \Rightarrow 3$ sign changes $\Rightarrow 3$ or 1 positive roots.
- Compute $f(-x) = (-x)^3 - 4(-x)^2 + 5(-x) - 2 = -x^3 - 4x^2 - 5x - 2$
- Signs: $- - - - \Rightarrow 0$ sign changes $\Rightarrow 0$ negative roots.

6.12 Sum of Coefficients

$f(1)$ gives the sum.

Example: $f(x) = x^3 - 2x^2 + x + 3$:

- $f(1) = 1^3 - 2(1)^2 + 1 + 3 = 1 - 2 + 1 + 3 = 3$

6.13 Vieta's Formulas

For $x^2 + ax + b = 0$: roots r_1, r_2,

$$r_1 + r_2 = -a,$$
$$r_1 r_2 = b.$$

Example: $x^2 - 5x + 6 = 0$

- Roots are 2 and 3.
- Sum = 2 + 3 = 5 = −(−5)
- Product = 2 * 3 = 6

For a polynomial of the form:

$$P(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$$

with roots r_1, r_2, r_3, r_4, Vieta's formulas give the following relationships:

$$\begin{aligned} r_1 + r_2 + r_3 + r_4 &= -a_1 \\ r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 &= a_2 \\ r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 &= -a_3 \\ r_1r_2r_3r_4 &= a_4 \end{aligned}$$

Example: $P(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

- Roots: $1, 2, 3, 4$
- Sum of roots: $1 + 2 + 3 + 4 = 10 = -(-10)$
- Sum of products of roots taken two at a time:

$$1 \cdot 2 + 1 \cdot 3 + 1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4 + 3 \cdot 4 = 2 + 3 + 4 + 6 + 8 + 12 = 35$$

- Sum of products of roots taken three at a time:

$$1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 4 + 1 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 = 6 + 8 + 12 + 24 = 50 = -(-50)$$

- Product of all roots: $1 \cdot 2 \cdot 3 \cdot 4 = 24$

Reciprocal Roots If roots of $f(x)$ are r, then roots of $x^n f(1/x)$ are $1/r$.

Example: $f(x) = x^2 - 3x + 2$, roots: 1 and 2 $\Rightarrow$ reciprocals: $\frac{1}{1}, \frac{1}{2}$.

6.14 Newton's Sums

Given a polynomial

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

with roots $r_1, r_2, \ldots, r_n$, define the **power sums** of the roots as

$$S_k = r_1^k + r_2^k + \cdots + r_n^k.$$

Newton's Sums (or Newton's Identities) provide a method to compute these sums S_k for any positive integer k using the coefficients of the polynomial without explicitly solving for the roots.

Newton's Sums Formulas:
For $k = 1, 2, \ldots, n$,

$$\begin{gathered} a_n S_1 + a_{n-1} = 0, \\ a_n S_2 + a_{n-1} S_1 + 2a_{n-2} = 0, \\ a_n S_3 + a_{n-1} S_2 + a_{n-2} S_1 + 3a_{n-3} = 0, \\ \vdots \\ a_n S_k + a_{n-1} S_{k-1} + \cdots + a_{n-k+1} S_1 + k a_{n-k} = 0. \end{gathered}$$

For $k > n$, the sums satisfy

$$a_n S_k + a_{n-1} S_{k-1} + \cdots + a_0 S_{k-n} = 0.$$

—

How to use Newton's Sums:

1. Ensure the polynomial is monic (if not, divide through by a_n).
2. Identify the coefficients $a_{n-1}, a_{n-2}, \ldots, a_0$.
3. Use the Newton's Sums formulas recursively to find $S_1, S_2, \ldots, S_k$.

—

Example:
Consider

$$f(x) = x^2 - 3x + 2,$$

which has roots r_1 and r_2.

Here,

$$a_2 = 1, \quad a_1 = -3, \quad a_0 = 2, \quad n = 2.$$

Step 1: Find $S_1 = r_1 + r_2$
Using Newton's Sum for $k = 1$:

$$a_2 S_1 + a_1 = 0 \implies 1 \cdot S_1 + (-3) = 0 \implies S_1 = 3.$$

Step 2: Find $S_2 = r_1^2 + r_2^2$
Using Newton's Sum for $k = 2$:

$$a_2 S_2 + a_1 S_1 + 2a_0 = 0,$$

substitute values:

$$1 \cdot S_2 + (-3)(3) + 2 \cdot 2 = 0 \implies S_2 - 9 + 4 = 0 \implies S_2 = 5.$$

Step 3: Find $S_3 = r_1^3 + r_2^3$
Since $k = 3 > n = 2$, use the formula for $k > n$:

$$a_2 S_3 + a_1 S_2 + a_0 S_1 = 0,$$

substitute values:

$$1 \cdot S_3 + (-3) \cdot 5 + 2 \cdot 3 = 0 \implies S_3 - 15 + 6 = 0 \implies S_3 = 9.$$

—

Summary:

$$\boxed{\begin{cases} S_1 = r_1 + r_2 = 3, \\ S_2 = r_1^2 + r_2^2 = 5, \\ S_3 = r_1^3 + r_2^3 = 9. \end{cases}}$$

Newton's Sums allow us to find sums of powers of roots using only the coefficients of the polynomial, which is especially useful when the roots are complicated or unknown.

6.15 Factoring Techniques

- **Sum and Difference of Cubes:**

  $$a^3 + b^3 = (a + b)(a^2 - ab + b^2), \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

  Example: $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

- **Factoring by Grouping:** Group terms and factor common binomials. **Example:** $x^3 + 3x^2 + 2x + 6 = (x + 3)(x^2 + 2)$

- **Difference of Squares:**

  $$a^2 - b^2 = (a - b)(a + b)$$

  Example: $x^2 - 16 = (x - 4)(x + 4)$

- **Quadratic-like:** Expressions that resemble a quadratic in form. **Example:** $x^4 - 5x^2 + 6 = (x^2 - 2)(x^2 - 3)$

- **Sophie Germain Identity:** Used to factor expressions of the form $a^4 + 4b^4$:

  $$a^4 + 4b^4 = (a^2 + 2b^2 - 2ab)(a^2 + 2b^2 + 2ab)$$

  Example: $x^4 + 4 = (x^2 - 2x + 2)(x^2 + 2x + 2)$

7 Exponents and Logarithms

7.1 Rational Exponents

Rational exponents express powers and roots in one notation:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Examples:

- $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$
- $16^{\frac{3}{4}} = (\sqrt[4]{16})^3 = 2^3 = 8$

Note: 0^0 is undefined unless explicitly defined in context.

7.2 Real Exponents

Real exponents extend the exponent rules beyond integers and rational numbers:

$$a^x \text{ is defined for all real } x > 0 \text{ when } a > 0$$

Examples:

- $2^{\sqrt{2}} \approx 2.6651$
- $10^{\pi} \approx 1385.45$

7.3 Exponential Functions

An exponential function has the form:

$$f(x) = a \cdot b^{x-h} + k$$

Where:

- a: vertical stretch/compression
- b: base $> 0, b \neq 1$
- h: horizontal shift
- k: vertical shift

Growth: $b > 1$
Decay: $0 < b < 1$
Example: Solve $2^{x+1} = 32$

$$2^{x+1} = 2^5 \Rightarrow x + 1 = 5 \Rightarrow x = 4$$

7.4 Logarithmic Functions

The inverse of an exponential function:

$$y = \log_b x \Leftrightarrow b^y = x$$

Where $b > 0, b \neq 1$.

Example:

$$\log_2 8 = 3 \text{ because } 2^3 = 8$$

Common logs:

- $\log x = \log_{10} x$
- $\ln x = \log_e x$

7.5 Properties of Logarithms

Basic Rules:

$$\begin{aligned}
\log_b(xy) &= \log_b x + \log_b y \\
\log_b\left(\frac{x}{y}\right) &= \log_b x - \log_b y \\
\log_b(x^k) &= k \log_b x \\
\log_b b = 1, &\quad \log_b 1 = 0
\end{aligned}$$

Examples:

- $\log_2(3x) = \log_2 3 + \log_2 x$
- $\log_3(x^4) = 4 \log_3 x$
- $\log_5\left(\frac{25}{x}\right) = \log_5 25 - \log_5 x = 2 - \log_5 x$

7.6 Change of Base Formula

To compute logs in another base:

$$\log_b x = \frac{\log_k x}{\log_k b} \quad (\text{any base } k > 0, k \neq 1)$$

Example:

$$\log_2 10 = \frac{\log_{10} 10}{\log_{10} 2} = \frac{1}{\log_{10} 2} \approx 3.3219$$

7.7 Applications

Simple Interest The total amount A after time t years is given by:

$$A = P(1 + rt)$$

where:

- P = principal (initial amount)
- r = annual interest rate (as a decimal)
- t = time in years

Example: If you invest \$1000 at 5% simple interest for 3 years:

$$A = 1000(1 + 0.05 \cdot 3) = 1000(1.15) = 1150$$

Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where:

- n = number of times interest is compounded per year

Example: If \$1000 is compounded quarterly at 6% for 2 years:

$$A = 1000\left(1 + \frac{0.06}{4}\right)^{4 \cdot 2} = 1000(1.015)^8 \approx 1000 \cdot 1.126 = 1126$$

Continuous Compounding

$$A = Pe^{rt}$$

Example: \$1000 at 5% interest compounded continuously for 4 years:

$$A = 1000e^{0.05 \cdot 4} = 1000e^{0.2} \approx 1000 \cdot 1.2214 = 1221.40$$

Exponential Growth

$$A = P(1 + r)^t$$

Example: A population of 1000 grows 3% per year:

$$A = 1000(1.03)^t$$

After 5 years:

$$A = 1000(1.03)^5 \approx 1000 \cdot 1.159 = 1159$$

Exponential Decay / Half-Life

$$A = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

where h is the half-life.

Example: A substance has a half-life of 3 years. Start with 200g:

$$A = 200\left(\frac{1}{2}\right)^{\frac{t}{3}}$$

After 6 years:

$$A = 200\left(\frac{1}{2}\right)^{2} = 200 \cdot \frac{1}{4} = 50$$

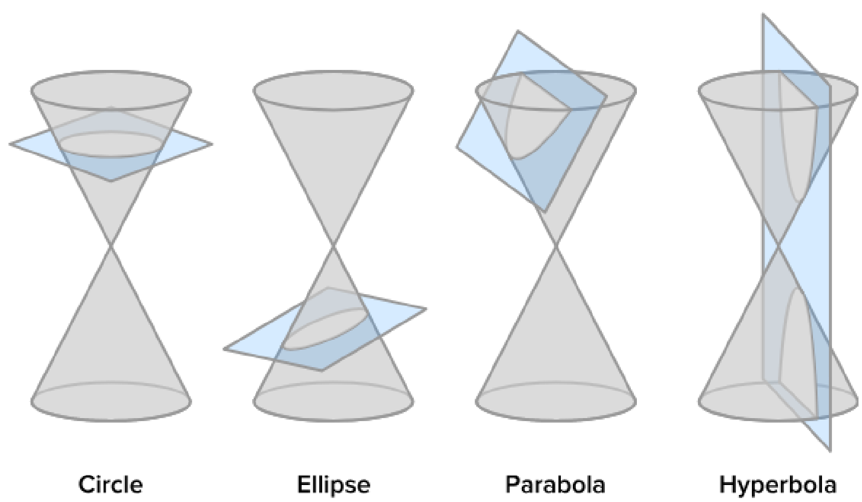
8 Conic Sections

8.1 Basics

A conic section is defined as a cross section of a double-napped cone and a plane.

Algebraically, the equation of a conic when graphed in the xy-plane is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$



Luckily, in Algebra 2, all conics we work with have no xy term; that is, $B = 0$. As shown in the above picture, different values of A and C give rise to four different conics.

8.2 Definitions

Each conic can be described by a locus definition. A locus is the set of all points that satisfy a relation. Here are the locus definitions for the four conics:

- **Circle:** A circle is the set of all points in a plane that are a fixed distance r (called the radius) from a fixed point C (called the center).
- **Parabola:** A parabola is the set of all points that are the same distance from a fixed point (called the focus) and a fixed line (called the directrix). The point halfway between the focus and the directrix is called the vertex. The line that passes through both the vertex and the focus, and is perpendicular to the directrix, is called the axis of symmetry.
- **Ellipse:** An ellipse is the set of all points such that the sum of the distances from two fixed points (called the foci) is constant. The longest line segment that passes through both foci and the center is called the major axis, and its endpoints are called the vertices. The shortest line segment through the center and perpendicular to the major axis is called the minor axis.
- **Hyperbola:** A hyperbola is the set of all points such that the absolute difference of the distances to two fixed points (called the foci) is constant. The midpoint between the foci is the center. The line segment connecting the two closest points on each branch (the vertices) is called the transverse axis. The line segment perpendicular to this one through the center is called the conjugate axis. A hyperbola also has two diagonal lines called asymptotes that the curve approaches but never touches.

8.3 Equations

Let (h, k) be the center (or vertex for parabolas). Below are the standard equations for each conic section with direction-specific notes and examples.

- **Circle:**
$$(x-h)^2 + (y-k)^2 = r^2$$
  This represents a circle with center (h, k) and radius r.

  Example: A circle with center $(2, -1)$ and radius 3 has equation:
$$(x-2)^2 + (y+1)^2 = 9$$

- **Parabola:**
  - *Opens Up/Down:*
$$(x-h)^2 = 4p(y-k)$$
    Opens upward if $p > 0$, downward if $p < 0$.

    Example (Upward): Vertex $(0, 0)$, $p = 2$:
$$x^2 = 8y$$

- *Opens Left/Right:*

$$(y-k)^2 = 4p(x-h)$$

  Opens right if $p > 0$, left if $p < 0$.

  Example (Leftward): Vertex $(1, -2)$, $p = -1$:

$$(y+2)^2 = -4(x-1)$$

- **Ellipse:**
  - *Major Axis Horizontal:*

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

    with $a > b$

    Example: Center $(0, 0)$, $a = 5$, $b = 3$:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

  - *Major Axis Vertical:*

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

    with $a > b$

    Example: Center $(2, 1)$, $a = 4$, $b = 2$:

$$\frac{(x-2)^2}{4} + \frac{(y-1)^2}{16} = 1$$

- **Hyperbola:**
  - *Opens Left/Right:*

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

    Example: Center $(0, 0)$, $a = 3$, $b = 2$:

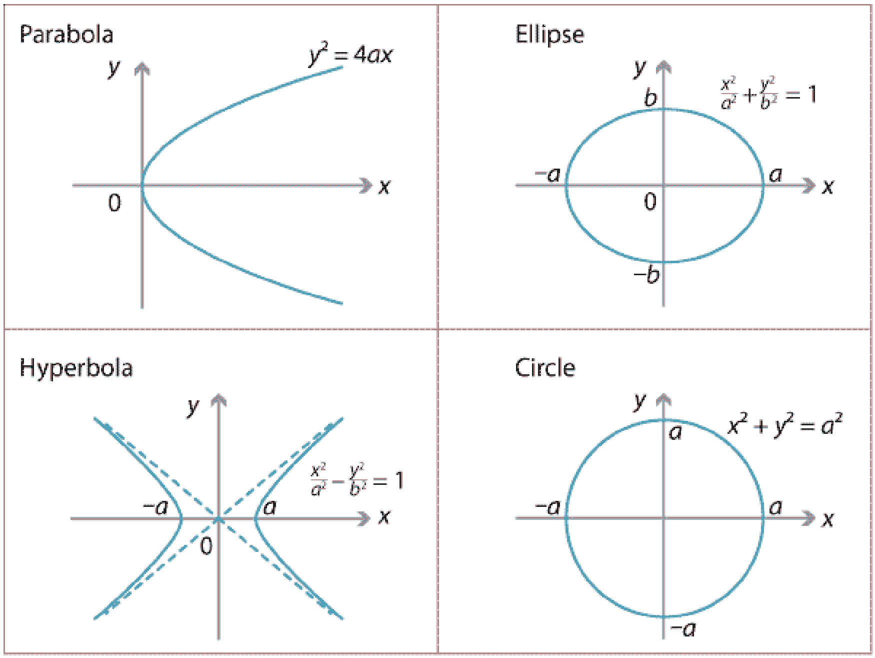
$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

  - *Opens Up/Down:*

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

    Example: Center $(1, -1)$, $a = 2$, $b = 1$:

$$\frac{(y+1)^2}{4} - \frac{(x-1)^2}{1} = 1$$



From the general form $Ax^2 + Cy^2 + Dx + Ey + F = 0$:

- If $A = C$, the conic is a circle.
- If $AC = 0$, the conic is a parabola.
- If $AC > 0$, the conic is an ellipse.
- If $AC < 0$, the conic is a hyperbola.

8.4 Orientation

- Parabolas: $x^2 = 4py$ opens up/down; $y^2 = 4px$ opens left/right.
- Ellipses: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a > b$, horizontal ellipse; if $b > a$, vertical ellipse.
- Hyperbolas: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ opens left/right; $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ opens up/down.

8.5 Foci

- **Ellipse:** $c^2 = a^2 - b^2$
  - If the major axis is horizontal: foci are at $(h \pm c, k)$
  - If the major axis is vertical: foci are at $(h, k \pm c)$
- **Hyperbola:** $c^2 = a^2 + b^2$
  - If the hyperbola opens left/right: foci are at $(h \pm c, k)$
  - If the hyperbola opens up/down: foci are at $(h, k \pm c)$

For parabola $x^2 = 4py$, focus is at $(0, p)$, directrix is the line $y = -p$.

8.6 Eccentricity

$$e = \frac{c}{a}$$

- Circle: $e = 0$
- Ellipse: $0 < e < 1$
- Parabola: $e = 1$
- Hyperbola: $e > 1$

8.7 Directrices

For ellipses and hyperbolas, the directrices are vertical or horizontal lines located at a distance of $\frac{a^2}{c}$ from the center. They are always parallel to the latus rectum of the conic.

- **Ellipse:**
  - If the major axis is horizontal: directrices are $x = h \pm \frac{a^2}{c}$ (vertical lines, parallel to the latus rectum).
  - If the major axis is vertical: directrices are $y = k \pm \frac{a^2}{c}$ (horizontal lines, parallel to the latus rectum).
- **Hyperbola:**
  - If the hyperbola opens left/right: directrices are $x = h \pm \frac{a^2}{c}$ (vertical lines, parallel to the latus rectum).
  - If the hyperbola opens up/down: directrices are $y = k \pm \frac{a^2}{c}$ (horizontal lines, parallel to the latus rectum).

8.8 Focal Parameter

The *focal parameter* is the distance from a focus to the corresponding directrix along the perpendicular.

- Parabola $x^2 = 4py$: focal parameter $= 2p$
- Ellipse: focal parameter $= \frac{b^2}{c}$
- Hyperbola: focal parameter $= \frac{b^2}{c}$

8.9 Latus Rectum

The *latus rectum* is the chord through a focus, perpendicular (or parallel) to the directrix.

- Parabola: length of latus rectum $= 4p$
- Ellipse: length of latus rectum $= \frac{2b^2}{a}$
- Hyperbola: length of latus rectum $= \frac{2b^2}{a}$
- Circle: length of latus rectum $= 2r$ (special case)

8.10 Miscellaneous

- Area of ellipse: πab
- Asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

$$y = \pm\frac{b}{a}x$$

  (shifted if center $\neq (0, 0)$)
- Degenerate conics:
  - Circle/Ellipse degenerate to a point or nothing (≤ 0)
  - Hyperbola degenerates to two intersecting lines ($= 0$)
  - Parabola degenerates to one or two parallel lines

9 Sequences and Series

9.1 Sequences

A **sequence** is an ordered list of numbers, denoted $\{a_n\}$, where n is a positive integer index.

Explicit formula:

Defines the n-th term directly as a function of n:

$$a_n = f(n)$$

Recursive formula:

Defines each term based on previous terms, with initial condition(s):

$$a_1 = \text{given}, \quad a_n = g(a_{n-1}, a_{n-2}, \ldots)$$

Example: Consider the sequence $2, 4, 6, 8, 10, \ldots$

Explicit formula:

$$a_n = 2n$$

which gives the term directly from its position n. For instance, $a_3 = 2 \times 3 = 6$.

Recursive formula:

$$a_1 = 2, \quad a_n = a_{n-1} + 2 \quad \text{for } n > 1$$

which defines each term based on the previous term plus 2. For example,

$$a_2 = a_1 + 2 = 2 + 2 = 4, \quad a_3 = a_2 + 2 = 4 + 2 = 6.$$

9.1.1 Arithmetic Sequence

A sequence where the difference between consecutive terms is constant:

$$a_n = a_1 + (n-1)d$$

where d is the common difference.

Recursive form:

$$a_1 = \text{given}, \quad a_n = a_{n-1} + d$$

Example: $5, 9, 13, 17, \ldots$ with $a_1 = 5$, $d = 4$, then

$$a_{10} = 5 + 9 \times 4 = 41$$

9.1.2 Geometric Sequence

A sequence where each term is multiplied by a constant ratio r:

$$a_n = a_1 \cdot r^{n-1}$$

Recursive form:

$$a_1 = \text{given}, \quad a_n = a_{n-1} \cdot r$$

Example: $3, 6, 12, 24, \ldots$, $a_1 = 3$, $r = 2$, then

$$a_5 = 3 \cdot 2^4 = 48$$

9.2 Series

A **series** is the sum of terms of a sequence:

$$S_n = \sum_{i=1}^{n} a_i$$

9.2.1 Arithmetic Series

Sum of an arithmetic sequence:

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}\big(2a_1 + (n-1)d\big)$$

Example: Sum of first 10 terms of $5, 9, 13, \ldots$

$$S_{10} = \frac{10}{2}(5 + 41) = 5 \times 46 = 230$$

9.2.2 Geometric Series

Sum of first n terms of a geometric sequence:

$$S_n = a_1 \frac{r^n - 1}{r - 1}, \quad r \neq 1$$

Infinite geometric series: (when $|r| < 1$)

$$S_\infty = \frac{a_1}{1 - r}$$

Example:

$$\sum_{n=1}^{\infty} 4 \left(\frac{1}{2}\right)^{n-1} = \frac{4}{1 - \frac{1}{2}} = 8$$

9.3 Summation Notation and Formulas

$$\sum_{i=1}^{n} a_i \quad \text{denotes the sum of terms } a_1 + a_2 + \cdots + a_n$$

Common formulas for sums of powers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

9.4 Arithmetic-Geometric Sequence

A sequence defined by multiplying an arithmetic term and a geometric term:

$$a_n = (a + (n-1)d) \cdot r^{n-1}$$

Sum of first n terms can be found using techniques involving geometric series and derivatives.

9.5 Telescoping Series

A series whose partial sums simplify due to cancellation of terms:

$$S_n = \sum_{i=1}^{n} (b_i - b_{i+1}) = b_1 - b_{n+1}$$

Example:

$$\sum_{n=1}^{5} \frac{1}{n(n+1)} = \sum_{n=1}^{5} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{6} = \frac{5}{6}$$

9.6 Logarithmic and Product Notation

Logarithmic sum:

$$\sum_{i=1}^{n} \log a_i = \log \left(\prod_{i=1}^{n} a_i \right)$$

Product notation:

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n$$

Example:

$$\prod_{i=1}^{4} i = 1 \times 2 \times 3 \times 4 = 24 = 4!$$

10 Combinatorics and Probability

10.1 Fundamental Counting Principle

If an event can occur in m ways and another independent event can occur in n ways, then the total number of ways both events can occur in sequence is:

$$m \times n$$

This principle extends to multiple events by multiplying the number of choices for each event.

Example: If there are 3 shirts and 4 pairs of pants, the number of outfit combinations is $3 \times 4 = 12$.

10.2 Permutations

A **permutation** is an arrangement of objects where order matters.

- Number of permutations of k objects chosen from n distinct objects:

$$P(n, k) = \frac{n!}{(n-k)!}$$

- Number of permutations of n distinct objects:

$$n! = n \times (n-1) \times \cdots \times 1$$

- **Linear permutations:** Arrangements in a line (order matters).

Permutations

Circular Permutations

Arrangements of n distinct objects around a circle, where rotations are considered the same arrangement, are given by:

$$\text{Number of circular permutations} = (n-1)!$$

Bracelet Permutations

When arranging n distinct objects in a circle to form a **bracelet**, not only are rotations considered the same, but **flips** (reflections) also produce identical arrangements. Therefore, the number of distinct bracelets is:

$$\text{Number of bracelet permutations} = \frac{(n-1)!}{2}$$

Example: Number of ways to arrange 3 people in a line from 5 people:

$$P(5,3) = \frac{5!}{2!} = 60$$

Example: Number of ways to seat 4 people around a round table:

$$(4-1)! = 3! = 6$$

Example: For the same 4 beads arranged on a bracelet (where flipping over the bracelet does not create a new arrangement), the number of distinct arrangements is:

$$\frac{(4-1)!}{2} = \frac{3!}{2} = \frac{6}{2} = 3$$

10.3 Combinations

A **combination** is a selection of objects where order does not matter.
- Number of combinations of k objects chosen from n:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Counting subsets of size k from a set of size n.
Example: Number of ways to choose a committee of 3 from 10 people:

$$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3! \cdot 7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = \frac{720}{6} = 120$$

10.4 Combinations and Products

When a selection requires choosing multiple groups independently, multiply combinations:
Example: Choosing 2 math books from 5 and 3 history books from 6:

$$\binom{5}{2} \times \binom{6}{3} = 10 \times 20 = 200$$

10.5 Mutually Exclusive and Independent Events

- **Mutually exclusive events:** Cannot occur simultaneously.

$$P(A \text{ and } B) = 0$$

- **Independent events:** Occurrence of one does not affect the probability of the other.

$$P(A \text{ and } B) = P(A) \times P(B)$$

Example: - Mutually exclusive: Rolling a die, getting 2 or 3 in a single roll (can't be both). - Independent: Flipping a coin and rolling a die.

10.6 Binomial Theorem

Expands powers of a binomial $(a+b)^n$ as a sum involving combinations:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Pascal's Triangle: A triangular array of binomial coefficients $\binom{n}{k}$ for $n \geq 0$, $0 \leq k \leq n$. Each number is the sum of the two numbers directly above it.

Example: Expand $(2x-3)^3$:

$$\begin{aligned}(2x-3)^3 &= \binom{3}{0}(2x)^3(-3)^0 + \binom{3}{1}(2x)^2(-3)^1 + \binom{3}{2}(2x)^1(-3)^2 + \binom{3}{3}(2x)^0(-3)^3 \\ &= 1 \cdot 8x^3 \cdot 1 - 3 \cdot 4x^2 \cdot 3 + 3 \cdot 2x \cdot 9 - 1 \cdot 1 \cdot 27 \\ &= 8x^3 - 36x^2 + 54x - 27\end{aligned}$$

10.7 Principle of Inclusion-Exclusion

Used to find the number of elements in the union of overlapping sets by correcting for overcounting:

For two sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For three sets A, B, C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

Example: In a group of 30 students, 18 study Math, 12 study Physics, and 8 study both. How many study Math or Physics?

$$|M \cup P| = 18 + 12 - 8 = 22$$

10.8 Conditional Probability

Probability of event X occurring given that event Y has occurred:

$$P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)} \quad \text{if } P(Y) > 0$$

Example: If 40% of students are female and 10% are female and play soccer, the probability a student plays soccer given they are female is:

$$P(\text{Soccer} \mid \text{Female}) = \frac{0.10}{0.40} = 0.25$$

10.9 Combination vs Permutation

- **Permutation:** Order matters. Number of ways to arrange k objects out of n:

$$P(n,k) = \frac{n!}{(n-k)!}$$

- **Combination:** Order does not matter. Number of ways to choose k objects out of n:

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example: Selecting 3 team leaders out of 5 people: - Permutations (order matters): $P(5,3) = 60$ - Combinations (order doesn't matter): $\binom{5}{3} = 10$

10.10 Paths on a Grid

Number of shortest paths from $(0,0)$ to (m,n) moving only right or up is:

$$\binom{m+n}{m} = \binom{m+n}{n}$$

Example: Number of ways to move from bottom-left corner to top-right corner of a 3 by 2 grid:

$$\binom{3+2}{3} = \binom{5}{3} = 10$$

10.11 Stars and Bars

Used to find the number of ways to distribute indistinguishable objects (stars) into distinguishable bins (bars).

Number of solutions in nonnegative integers to:

$$x_1 + x_2 + \cdots + x_k = n$$

is

$$\binom{n+k-1}{k-1}$$

Example: Number of ways to put 5 identical candies into 3 distinct bags:

$$\binom{5+3-1}{3-1} = \binom{7}{2} = 21$$

10.12 Geometric Probability

Probability is the ratio of the favorable geometric measure (length, area, volume) to the total measure of the sample space.

$$P = \frac{\text{Measure of favorable region}}{\text{Measure of total region}}$$

Example: Randomly pick a point on a line segment of length 10. Probability the point lies within the first 3 units:

$$P = \frac{3}{10} = 0.3$$

11 Matrices

11.1 Matrix Operations

- **Addition:** Matrices must have the same dimensions. Add corresponding elements.

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 3+7 \\ 2+6 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 8 & 12 \end{bmatrix}$$

- **Multiplication:** For $A_{m\times n}$ and $B_{n\times p}$, multiply rows of A by columns of B.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1\cdot 5 + 2\cdot 7 & 1\cdot 6 + 2\cdot 8 \\ 3\cdot 5 + 4\cdot 7 & 3\cdot 6 + 4\cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

- **Scalar Multiplication:** Multiply each element by the scalar.

$$3 \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

- **Identity Matrix I_n:** Square matrix with 1's on the diagonal and 0's elsewhere. Acts as multiplicative identity.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad AI = IA = A$$

11.2 Determinants

For how to compute, click to go to the Determinants section

Properties:

- $\det(AB) = \det(A)\det(B)$
- $\det(I) = 1$
- A matrix is invertible if and only if $\det(A) \neq 0$

11.3 Square Matrices and Inverses

Square Matrix: A matrix A is called **square** if it has the same number of rows and columns, i.e., A is $n \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Only square matrices can have inverses.

Invertible Matrix: A square matrix A is **invertible** (or **nonsingular**) if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the $n \times n$ identity matrix.

Invertibility Condition: A is invertible if and only if $\det(A) \neq 0$.

11.4 Inverse of an $n \times n$ Matrix

Adjugate (Classical) Formula:

For an invertible $n \times n$ matrix A,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

where $\operatorname{adj}(A)$ is the **adjugate matrix**, defined as the transpose of the cofactor matrix:

$$\operatorname{adj}(A) = [C_{ji}]^T$$

Each cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix obtained by deleting the i-th row and j-th column of A.

Example: Compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Step 1: Compute the cofactor matrix C:

$$C = \begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} (4)(6) - (5)(0) & -(0 \cdot 6 - 5 \cdot 1) & (0 \cdot 0 - 4 \cdot 1) \\ -(2 \cdot 6 - 3 \cdot 0) & (1 \cdot 6 - 3 \cdot 1) & -(1 \cdot 0 - 2 \cdot 1) \\ (2 \cdot 5 - 3 \cdot 4) & -(1 \cdot 5 - 3 \cdot 0) & (1 \cdot 4 - 2 \cdot 0) \end{bmatrix}$$

$$C = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}$$

Step 2: Transpose the cofactor matrix to get $\text{adj}(A)$:

$$\text{adj}(A) = C^T = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

Step 3: Compute $\det(A)$ (use cofactor expansion along the first row):

$$\det(A) = 1 \cdot \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 2 \cdot \begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} + 3 \cdot \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = 1(24) - 2(-5) + 3(-4) = 24 + 10 - 12 = 22$$

Step 4: Multiply $\frac{1}{\det(A)}$ with $\text{adj}(A)$:

$$A^{-1} = \frac{1}{22} \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$$

—

Final Answer:

$$A^{-1} = \begin{bmatrix} \frac{12}{11} & -\frac{6}{11} & -\frac{1}{11} \\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22} \\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11} \end{bmatrix}$$

- Only square matrices can have inverses.
- A is invertible if and only if $\det(A) \neq 0$.
- For 2×2, inverse has a simple formula.

11.5 Solving Linear Systems

Express system as $A\vec{x} = \vec{b}$.

If A is invertible:

$$\vec{x} = A^{-1}\vec{b}$$

Example:

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \end{cases} \Rightarrow A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

Calculate A^{-1} and then $\vec{x} = A^{-1}\vec{b}$.