FAMAT Algebra 1 Study Guide

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Contents

1 Foundations (Pre-Algebra Review)

1.1 Numbers & Operations

Types of Numbers

- **Natural Numbers:** $1, 2, 3, 4, \ldots$
  These are the "counting numbers" used in everyday life.

- **Whole Numbers:** $0, 1, 2, 3, 4, \ldots$
  This set is the same as the natural numbers, but also includes 0.

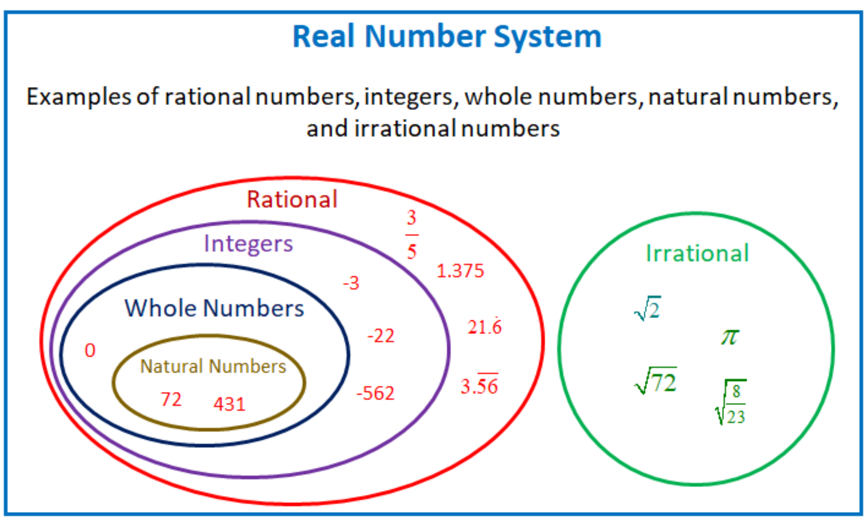
- **Integers:** $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
  Integers include all positive and negative whole numbers, as well as 0.

- **Rational Numbers:** Numbers that can be written as a fraction $\frac{p}{q}$ where p and q are integers and $q \neq 0$. Examples: $-3,\ \frac{2}{5},\ 4.25,\ 0.333\ldots$

- **Irrational Numbers:** Numbers that cannot be written as a fraction. Their decimal expansions are non-repeating and non-terminating. Examples: $\pi,\ \sqrt{2},\ e$.

- **Real Numbers:** The set of all rational and irrational numbers combined. This includes every point on the number line.

Visual: Number System Diagram



Practice

Classify the following numbers:

1. -3 (Integer, Rational, Real)
2. 2.5 (Rational, Real)

3. 16 (Natural, Whole, Integer, Rational, Real)

4. π (Irrational, Real)

1.2 Order of Operations (PEMDAS)

Rule

To evaluate mathematical expressions, follow the correct order:

$$\begin{aligned} \text{Parentheses} &\rightarrow \text{Exponents} \\ &\rightarrow \text{Multiplication/Division (left to right)} \\ &\rightarrow \text{Addition/Subtraction (left to right)} \end{aligned}$$

This is often remembered with the acronym **PEMDAS**.

Example

Simplify:

$$3 + 4 \times 2^2$$

Step 1: Exponents: $2^2 = 4$. Expression becomes $3 + 4 \times 4$.
Step 2: Multiplication: $4 \times 4 = 16$. Expression becomes $3 + 16$.
Step 3: Addition: $3 + 16 = 19$.

Common Mistake

Do not always do multiplication before division or addition before subtraction. Instead, do multiplication & division in the order they appear from **left to right**, and the same for addition & subtraction.

Example: $20 \div 5 \times 2 = (20 \div 5) \times 2 = 4 \times 2 = 8$ (not $20 \div (5 \times 2)$).

1.3 Absolute Value

Definition

The absolute value of a number a, written $|a|$, is its distance from zero on the number line. Distance is always non-negative.

$$|5| = 5 \quad \text{and} \quad |-5| = 5$$

Algebraic Rule

$$|x| = a \implies x = \pm a$$

That means if the absolute value of x equals a positive number a, then x could be either a or $-a$.

Examples

- $|7| = 7$
- $|-12| = 12$
- Solve $|x| = 4$: $x = 4$ or $x = -4$.

Practice

Solve $|x - 3| = 5$.
This means the distance between x and 3 is 5. So:

$$x - 3 = 5 \quad \text{or} \quad x - 3 = -5$$

$$x = 8 \quad \text{or} \quad x = -2$$

2 Algebraic Expressions & Equations

2.1 Translating Words into Algebra

Vocabulary

- **Sum:** addition
- **Difference:** subtraction
- **Product:** multiplication
- **Quotient:** division
- **"More than":** addition
- **"Less than":** subtraction

Examples

- "5 more than twice a number" $\rightarrow 2x + 5$
- "3 less than a number" $\rightarrow x - 3$
- "The product of 4 and a number" $\rightarrow 4x$
- "The quotient of a number and 7" $\rightarrow \frac{x}{7}$

Practice

Translate these into algebra:

1. 7 more than a number **Answer:** $x + 7$
2. Twice a number minus 5 **Answer:** $2x - 5$
3. 10 less than three times a number **Answer:** $3x - 10$

2.2 Properties of Real Numbers

Key Properties

- **Commutative:** $a + b = b + a$, $ab = ba$
- **Associative:** $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$
- **Distributive:** $a(b + c) = ab + ac$
- **Identity:** $a + 0 = a$, $a \cdot 1 = a$
- **Inverse:** $a + (-a) = 0$, $a \cdot \frac{1}{a} = 1$ (for $a \neq 0$)

Example: Simplify an Expression

$$3(x + 4) = 3 \cdot x + 3 \cdot 4 = 3x + 12$$

Practice

Simplify each expression:

1. $2(x + 5)$ **Answer:** $2x + 10$
2. $4(y - 3)$ **Answer:** $4y - 12$
3. $5(a + 2b)$ **Answer:** $5a + 10b$

2.3 Solving Linear Equations (1 Variable)

Strategy: Inverse Operations / Balance Method

To solve $ax + b = c$:

1. Undo addition/subtraction first
2. Undo multiplication/division second
3. Always perform the same operation on both sides

Solve $2x + 5 = 13$:

$$2x + 5 - 5 = 13 - 5 \quad \Rightarrow \quad 2x = 8$$

$$2x/2 = 8/2 \quad \Rightarrow \quad x = 4$$

Word Problem

A number plus 7 equals 15. Find the number.

$$x + 7 = 15 \quad \Rightarrow \quad x = 8$$

Practice Problems

1. $3x - 4 = 11$ **Answer:** $x = 5$
2. $5 + 2x = 17$ **Answer:** $x = 6$
3. $7x + 9 = 30$ **Answer:** $x = 3$

2.4 Linear Inequalities

Rules

- Solve inequalities like equations.
- **Important:** When multiplying or dividing by a negative number, **flip the inequality sign**.

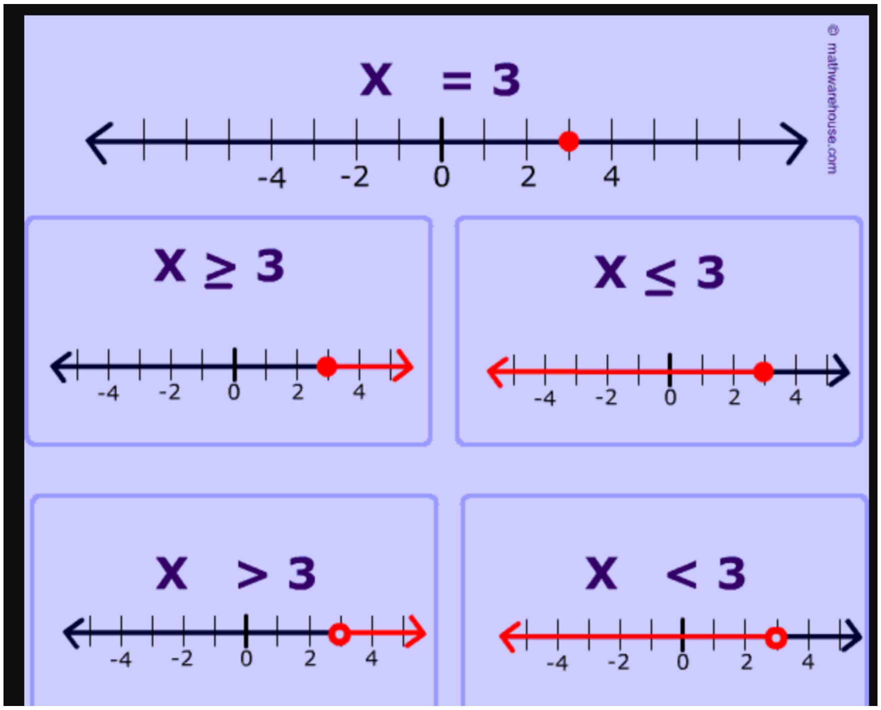
Example 1

Solve $-2x + 3 < 7$:

$$-2x + 3 - 3 < 7 - 3 \quad \Rightarrow \quad -2x < 4$$

$$x > -2 \quad \text{(flip the inequality because we divided by -2)}$$

Graphing Solution on a Number Line



Practice Problems

1. $3x - 5 \leq 10$ **Answer:** $x \leq 5$
2. $-4x + 2 > 6$ **Answer:** $x < -1$ (flipped inequality because of dividing by negative)
3. $2x + 3 < 9$ **Answer:** $x < 3$

3 Linear Functions and Graphs

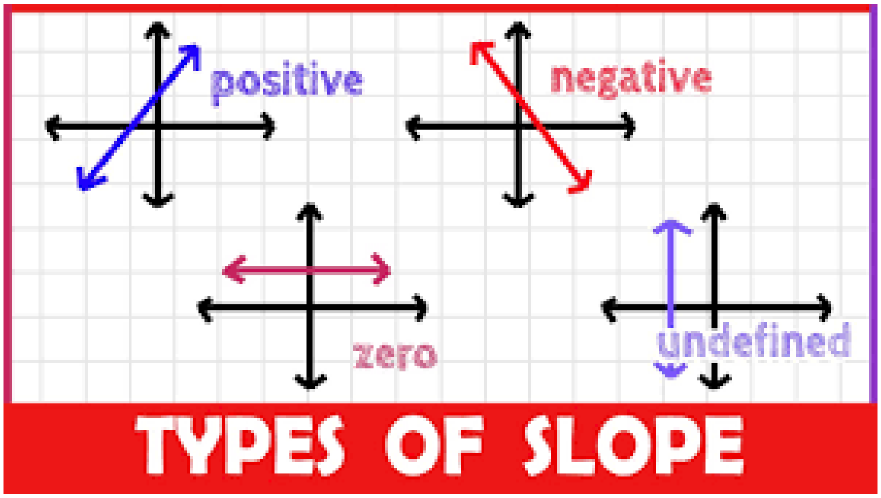
3.1 Slope and Rate of Change

Definition: Slope measures how steep a line is:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x}$$

Types of slopes:

- Positive: $m > 0$ (line rises left to right)
- Negative: $m < 0$ (line falls left to right)
- Zero: $m = 0$ (horizontal line)
- Undefined: vertical line



Example: Line through $(2, 3)$ and $(5, 9)$:

$$m = \frac{9-3}{5-2} = \frac{6}{3} = 2$$

3.2 Equations of Lines

Slope-intercept form:

$$y = mx + b$$

Example: $m = 2$, $b = 3$

$$y = 2x + 3$$

Point-slope form:

$$y - y_1 = m(x - x_1)$$

Example: $m = -1$, point $(4, 2)$

$$y - 2 = -1(x - 4) \implies y = -x + 6$$

Standard Form:

$$Ax + By = C, \quad \text{where } A, B, C \text{ are integers and } A > 0$$

Example: Convert $y = 2x + 5$ to standard form:

$$y = 2x + 5 \implies -2x + y = 5 \implies 2x - y = -5$$

Note: We multiply through by -1 to make A positive, as required by the convention.

3.3 Parallel and Perpendicular Lines

Parallel Lines: Two lines are parallel if they have the same slope.

$$m_1 = m_2$$

Example: $y = 3x + 1$ and $y = 3x - 4$ are parallel because both have slope 3.

Perpendicular Lines: Two lines are perpendicular if their slopes are **negative reciprocals** of each other.

Reciprocal: The reciprocal of a number $\frac{a}{b}$ is $\frac{b}{a}$.

Negative Reciprocal: Take the reciprocal and change its sign.

$$m_1 \cdot m_2 = -1$$

Example: $y = 2x + 3$ has slope 2. A line perpendicular to it must have slope $-\frac{1}{2}$ (negative reciprocal of 2).

Check: $2 \cdot \left(-\frac{1}{2}\right) = -1$, so $y = -\frac{1}{2}x + 1$ is perpendicular.

3.4 Piecewise Functions

Definition: A **piecewise function** is a function that is defined by different expressions (formulas) for different parts of its domain. Each formula applies to a specific interval of input values (x).

$$f(x) = \begin{cases} x + 2, & x < 3 \\ 2x - 1, & x \geq 3 \end{cases}$$

How it works:

- Look at the input value x.
- Determine which interval x belongs to.
- Use the corresponding formula for that interval to calculate $f(x)$.

Example: Evaluate $f(2)$ and $f(4)$:

- $f(2)$: Since $2 < 3$, use the first formula $f(x) = x + 2$:

$$f(2) = 2 + 2 = 4$$

- $f(4)$: Since $4 \geq 3$, use the second formula $f(x) = 2x - 1$:

$$f(4) = 2(4) - 1 = 8 - 1 = 7$$

Tip: Always check which interval the input belongs to. Using the wrong formula is a common mistake.

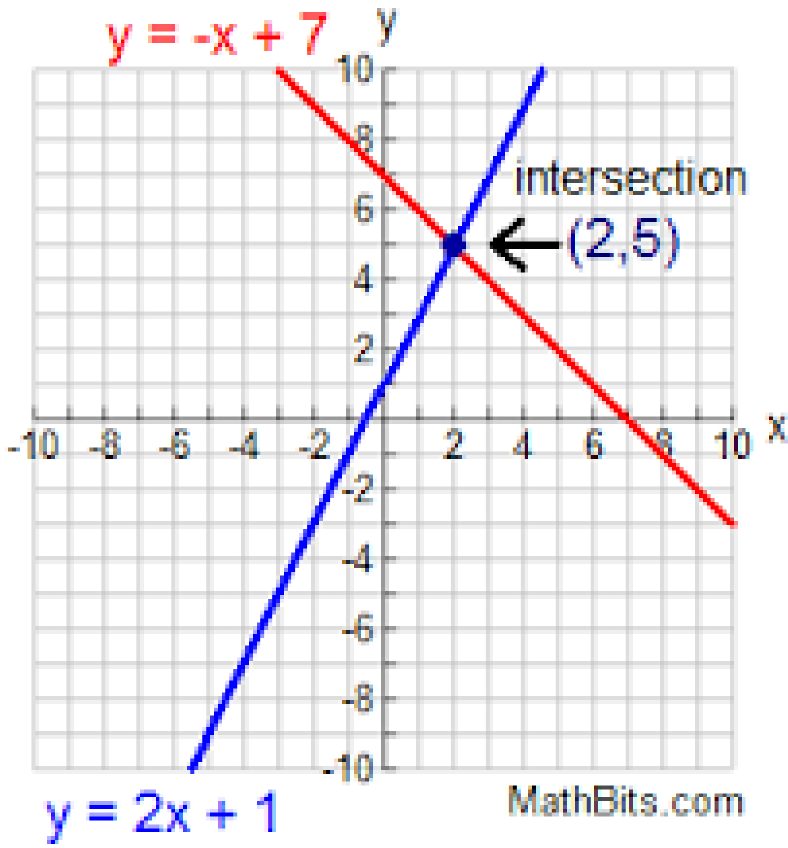
4 Systems of Equations

4.1 Solving Systems of Equations

A **system of equations** consists of two or more equations with the same variables. The goal is to find values of the variables that satisfy all equations simultaneously.

Methods for Solving

1. Graphing Method: - Graph each equation on the same coordinate plane. - The point(s) where the lines intersect is the solution.



2. Substitution Method: - Solve one equation for one variable and substitute into the other equation.

Example: Solve

$$x + y = 5, \quad y = 2x$$

Substitute $y = 2x$ into $x + y = 5$:

$$x + 2x = 5 \implies 3x = 5 \implies x = \frac{5}{3}$$

Then $y = 2\left(\frac{5}{3}\right) = \frac{10}{3}$ **Solution:** $\left(\frac{5}{3}, \frac{10}{3}\right)$

3. Elimination (Addition) Method: - Add or subtract equations to eliminate one variable.

Example: Solve

$$2x + 3y = 12, \quad 4x - 3y = 6$$

Add equations:

$$(2x + 3y) + (4x - 3y) = 12 + 6 \implies 6x = 18 \implies x = 3$$

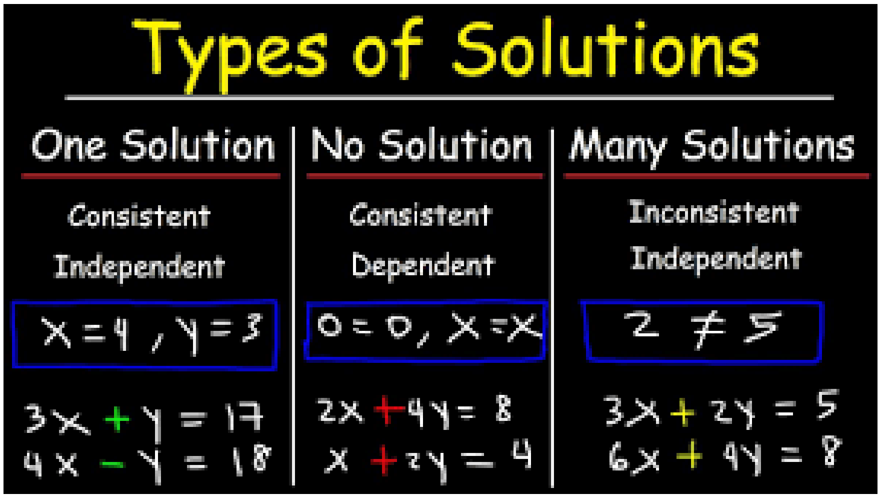
Substitute $x = 3$ into $2x + 3y = 12$:

$$2(3) + 3y = 12 \implies 6 + 3y = 12 \implies 3y = 6 \implies y = 2$$

Solution: $(3, 2)$

Types of Solutions

- **One solution:** lines intersect at a single point.
- **No solution:** lines are parallel (same slope, different y-intercepts).
- **Infinitely many solutions:** lines are the same (same slope and y-intercept).



4.2 Applications: Word Problems

1. Mixture Problems: *Example:* A chemist needs 10 liters of a solution that is 30% acid. She has a 20% solution and a 50% solution. How much of each should she mix?

Let x = liters of 20% solution, y = liters of 50% solution.

$$x + y = 10 \quad \text{(total solution)}$$

$$0.2x + 0.5y = 0.3 \cdot 10 = 3 \quad \text{(acid content)}$$

Solve by substitution or elimination:

$$y = 10 - x$$

$$0.2x + 0.5(10 - x) = 3 \implies 0.2x + 5 - 0.5x = 3 \implies -0.3x = -2 \implies x = \frac{20}{3} \approx 6.67$$

$$y = 10 - 6.67 = 3.33$$

Answer: Mix approximately 6.67 L of 20% solution and 3.33 L of 50% solution.

2. Work Problems: *Example:* Worker A can complete a job in 6 hours, Worker B in 4 hours. How long if they work together?

Let x = fraction of job per hour by A, y = fraction by B:

$$x = \frac{1}{6}, \quad y = \frac{1}{4}, \quad \text{together: } x + y = \frac{1}{T}$$

$$\frac{1}{6} + \frac{1}{4} = \frac{1}{T} \implies \frac{2}{12} + \frac{3}{12} = \frac{5}{12} = \frac{1}{T} \implies T = \frac{12}{5} = 2.4 \text{ hours}$$

Answer: They finish the job in 2.4 hours.

3. Rate/Distance Problems: *Example:* A boat travels 20 km downstream in 2 hours and returns upstream in 4 hours. Find the speed of the boat in still water and the speed of the current.

Let b = boat speed (km/h), c = current speed (km/h):

$$b + c = \frac{20}{2} = 10, \quad b - c = \frac{20}{4} = 5$$

$$b + c = 10, \quad b - c = 5 \implies 2b = 15 \implies b = 7.5$$

$$7.5 + c = 10 \implies c = 2.5$$

Answer: Boat speed = 7.5 km/h, Current speed = 2.5 km/h.

4. Coins Problems: *Example:* A jar contains nickels and dimes totaling \$2.40. There are 24 coins. How many of each coin?

Let n = nickels, d = dimes:

$$n + d = 24, \quad 0.05n + 0.10d = 2.40$$

Multiply the second equation by 100:

$$5n + 10d = 240$$

Solve:

$$n + d = 24 \implies n = 24 - d$$

$$5(24 - d) + 10d = 240 \implies 120 - 5d + 10d = 240 \implies 5d = 120 \implies d = 24$$

$$n = 24 - 24 = 0$$

Answer: 0 nickels, 24 dimes.

5. Current/Wind Problems: *Example:* A plane flies 300 km with a tailwind in 1 hour and returns against the wind in 1.5 hours. Find the plane's speed in still air and wind speed.

Let p = plane speed, w = wind speed:

$$p + w = \frac{300}{1} = 300, \quad p - w = \frac{300}{1.5} = 200$$

$$p + w = 300, \quad p - w = 200 \implies 2p = 500 \implies p = 250$$

$$250 + w = 300 \implies w = 50$$

Answer: Plane speed = 250 km/h, Wind speed = 50 km/h.

5 Polynomials & Factoring

5.1 Polynomials

Definition: A **polynomial** is an expression made up of terms that are constants, variables, or products of constants and variables raised to non-negative integer powers.

$$\text{Example: } 3x^4 - 5x^2 + 2x - 7$$

Components of a Polynomial:

- **Degree:** The highest power of the variable. **Example:** $3x^4 - 5x^2 + 2x - 7$ has degree 4.

- **Leading Coefficient:** The coefficient of the term with the highest degree. **Example:** The leading coefficient of $3x^4 - 5x^2 + 2x - 7$ is 3.

Adding and Subtracting Polynomials

Combine **like terms** (terms with the same variable and exponent).

Example:

$$(3x^2 + 5x - 2) + (2x^2 - 3x + 7) = 5x^2 + 2x + 5$$

Multiplying Polynomials (Step-by-Step)

To multiply polynomials, each term in one polynomial must be multiplied by each term in the other.

1. Multiplying Binomials: FOIL Method

FOIL stands for: First, Outer, Inner, Last. It is a shortcut for multiplying two binomials.

Example: Multiply $(x + 3)(x + 5)$ using FOIL:

- **First:** Multiply the first terms in each binomial: $x \cdot x = x^2$

- **Outer:** Multiply the outer terms: $x \cdot 5 = 5x$

- **Inner:** Multiply the inner terms: $3 \cdot x = 3x$

- **Last:** Multiply the last terms: $3 \cdot 5 = 15$

Now combine all terms:

$$x^2 + 5x + 3x + 15 = x^2 + 8x + 15$$

2. Multiplying a Binomial by a Trinomial (Distributive Property)
Example: Multiply $(x + 2)(x^2 + 3x + 4)$
Step 1: Distribute x to each term in the trinomial:

$$x(x^2 + 3x + 4) = x^3 + 3x^2 + 4x$$

Step 2: Distribute 2 to each term in the trinomial:

$$2(x^2 + 3x + 4) = 2x^2 + 6x + 8$$

Step 3: Add the results together:

$$x^3 + 3x^2 + 4x + 2x^2 + 6x + 8$$

Step 4: Combine like terms:

$$x^3 + (3x^2 + 2x^2) + (4x + 6x) + 8 = x^3 + 5x^2 + 10x + 8$$

Tip: Always combine like terms at the end to simplify the polynomial.

Laws of Exponents

- $x^a \cdot x^b = x^{a+b}$
- $\frac{x^a}{x^b} = x^{a-b}$, $x \neq 0$
- $(x^a)^b = x^{ab}$
- $(xy)^a = x^a y^a$

5.2 Factoring Polynomials

Definition: Factoring is rewriting a polynomial as a product of simpler polynomials or constants.

1. Factoring out the Greatest Common Factor (GCF)

Steps: 1. Identify the largest factor common to all terms. 2. Factor it out.
Example:

$$6x^3 + 9x^2 = 3x^2(2x + 3)$$

2. Factoring Trinomials ($x^2 + bx + c$)

Goal: Find two numbers m and n such that:

$$m \cdot n = c, \quad m + n = b$$

Example: Factor $x^2 + 5x + 6$:

Numbers that multiply to 6 and add to 5 are 2 and 3.

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

3. Difference of Squares

$$a^2 - b^2 = (a - b)(a + b)$$

Example:

$$x^2 - 16 = (x - 4)(x + 4)$$

4. Solving Quadratic Equations by Factoring

Steps: 1. Factor the quadratic. 2. Set each factor equal to zero. 3. Solve for x.

Example: Solve $x^2 + 5x + 6 = 0$

Factor: $x^2 + 5x + 6 = (x + 2)(x + 3)$

Set each factor equal to zero:

$$x + 2 = 0 \implies x = -2, \quad x + 3 = 0 \implies x = -3$$

5.3 Polynomial Division

1. Polynomial Long Division (Linear Divisors Only)

Definition: Polynomial long division is similar to regular long division with numbers. You divide the highest-degree term first, then subtract, bring down the next term, and repeat until all terms are used.

Example: Divide $x^2 + 5x + 6$ by $x + 2$

Step 1: Divide the first term of the dividend by the first term of the divisor.

$$\frac{x^2}{x} = x$$

This is the first term of the quotient.

Step 2: Multiply the entire divisor by this term.

$$x \cdot (x + 2) = x^2 + 2x$$

Step 3: Subtract this result from the dividend.

$$(x^2 + 5x + 6) - (x^2 + 2x) = 3x + 6$$

Step 4: Repeat the process with the new polynomial. Divide the first term of the new polynomial by the first term of the divisor:

$$\frac{3x}{x} = 3$$

Multiply the divisor by 3:

$$3(x + 2) = 3x + 6$$

Subtract:

$$(3x + 6) - (3x + 6) = 0$$

Step 5: Write the final quotient and remainder.

$$\text{Quotient: } x + 3, \quad \text{Remainder: } 0$$

Check: Multiply quotient by divisor: $(x + 3)(x + 2) = x^2 + 5x + 6$. Correct!

2. Synthetic Division (Linear Divisors Only)

Definition: Synthetic division is a shortcut method for dividing polynomials when the divisor is linear $(x - k)$. It uses only coefficients and is faster than long division.

Example: Divide $x^2 + 5x + 6$ by $x + 2$

Step 1: Write coefficients of the dividend.

$$1 \quad 5 \quad 6$$

(Corresponding to x^2, x, and the constant.)

Step 2: Use the zero of the divisor. For $x + 2$, the zero is -2 (because $x + 2 = 0 \implies x = -2$).

Step 3: Synthetic division table.

$$\begin{array}{r|rrr} -2 & 1 & 5 & 6 \\ & & -2 & -6 \\ \hline & 1 & 3 & 0 \end{array}$$

Explanation of steps:

- Bring down the first coefficient: 1
- Multiply it by -2 (the zero of the divisor) and write under the next coefficient: $1 \cdot -2 = -2$
- Add down the column: $5 + (-2) = 3$
- Multiply $3 \cdot -2 = -6$, write under next coefficient
- Add: $6 + (-6) = 0$ (remainder)

Step 4: Write the quotient and remainder. The numbers at the bottom row, except the remainder, are the coefficients of the quotient: $1x + 3$, remainder 0.

Check: $(x + 2)(x + 3) + 0 = x^2 + 5x + 6$. Correct!

6 Quadratic Functions

6.1 Objective

Learn how to solve and graph quadratic equations, and understand how their solutions relate to real-world situations.

6.2 What is a Quadratic Equation?

A quadratic equation is any equation that can be written in the form:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

- a, b, c are numbers (coefficients)
- x is the variable

6.3 How to Solve Quadratics

1. Factoring: Break the quadratic into two simpler expressions multiplied together. Then set each equal to zero.

Example:

$$x^2 + 5x + 6 = 0$$

Find two numbers that multiply to 6 and add to 5: 2 and 3.

$$x^2 + 5x + 6 = (x + 2)(x + 3) = 0$$

Set each factor to zero: $x + 2 = 0 \implies x = -2$, $x + 3 = 0 \implies x = -3$

2. Quadratic Formula: Use when factoring is difficult.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

6.4 Graphing Quadratics

- Standard form: $y = ax^2 + bx + c$
- Vertex form: $y = a(x - h)^2 + k$, where (h, k) is the vertex

Vertex: The highest or lowest point of the parabola (depending on a)

Axis of Symmetry: Vertical line through the vertex. It divides the parabola into two mirror-image halves.

- **Vertex form:** If $y = a(x - h)^2 + k$, then the axis of symmetry is

$$x = h$$

- **Standard form:** If $y = ax^2 + bx + c$, then the axis of symmetry is

$$x = -\frac{b}{2a}$$

Examples:

- Vertex form: $y = 2(x-3)^2 + 5$ Vertex: $(3, 5)$, Axis of symmetry: $x = 3$
- Standard form: $y = x^2 + 6x + 5$ Axis of symmetry: $x = -\frac{6}{2(1)} = -3$, Vertex: $(-3, -4)$

6.5 Completing the Square

Completing the square is a method to rewrite a quadratic in vertex form:

$$y = a(x-h)^2 + k$$

This makes it easy to identify the vertex and graph the parabola.

Step-by-Step Example:
Rewrite the quadratic:

$$y = x^2 + 6x + 5$$

1. **Group the x^2 and x terms:**

$$y = (x^2 + 6x) + 5$$

2. **Take half of the coefficient of x and square it:** Coefficient of x is 6. Half of 6 is 3. Square it: $3^2 = 9$.

3. **Add and subtract this square inside the parentheses:**

$$y = (x^2 + 6x + 9 - 9) + 5$$

4. **Rewrite the perfect square trinomial as a squared binomial:**

$$y = ((x+3)^2 - 9) + 5$$

5. **Combine constants:**

$$y = (x+3)^2 - 4$$

Result: Vertex form: $y = (x+3)^2 - 4$ **Vertex:** $(-3, -4)$

Why it works: We are essentially creating a perfect square trinomial inside the quadratic so that it factors neatly. This shows the vertex directly and makes graphing easier.

7 Rational and Radical Expressions

7.1 Rational Expressions

Definition: A rational expression is a fraction in which the numerator and/or denominator is a polynomial.

Simplifying Fractions

1. Factor the numerator and denominator completely.
2. Cancel any common factors (except zero).

Example: Simplify

$$\frac{x^2 - 4}{x^2 + 3x + 2}$$

1. Factor numerator and denominator:

$$\frac{(x-2)(x+2)}{(x+1)(x+2)}$$

2. Cancel the common factor $(x+2)$:

$$\frac{x-2}{x+1}$$

Adding/Subtracting Fractions

1. Find the least common denominator (LCD). 2. Rewrite each fraction with the LCD. 3. Combine the numerators. 4. Simplify if possible.

Example:

$$\frac{x}{x^2-4} + \frac{2}{x+2}$$

1. Factor denominators: $x^2 - 4 = (x-2)(x+2)$

$$\frac{x}{(x-2)(x+2)} + \frac{2}{x+2}$$

2. Rewrite second fraction with LCD:

$$\frac{x}{(x-2)(x+2)} + \frac{2(x-2)}{(x-2)(x+2)}$$

3. Combine numerators:

$$\frac{x+2(x-2)}{(x-2)(x+2)} = \frac{3x-4}{(x-2)(x+2)}$$

Multiplication and Division

- **Multiplication:** Multiply numerators and denominators separately, then simplify. Example: $\frac{x}{x+1} \cdot \frac{x+1}{x-2} = \frac{x}{x-2}$
- **Division:** Multiply by the reciprocal of the second fraction. Example: $\frac{x}{x+1} \div \frac{x-2}{x+1} = \frac{x}{x+1} \cdot \frac{x+1}{x-2} = \frac{x}{x-2}$

Domain: Denominator $\neq 0$. For $\frac{3x-4}{(x-2)(x+2)}$, $x \neq 2, -2$.
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7.2 Radical Expressions

Definition: Expressions containing roots (square roots $\sqrt{x}$, cube roots $\sqrt[3]{x}$, etc.)

Simplifying Radicals

1. Factor the number under the radical into perfect squares (or cubes for cube roots).
2. Take the square root of perfect squares out of the radical.

Example:

$$\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$$

Operations with Radicals

- **Adding/Subtracting:** Only like radicals can be combined. Example: $5\sqrt{2} + 3\sqrt{2} = 8\sqrt{2}$
- **Multiplying:** Multiply coefficients and radicands separately. Example: $\sqrt{2} \cdot \sqrt{8} = \sqrt{16} = 4$
- **Dividing and Rationalizing:** Multiply numerator and denominator to remove radicals from the denominator. Example: $\frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

Connection to Exponents

$$x^{1/2} = \sqrt{x}, \quad x^{1/3} = \sqrt[3]{x}, \quad x^{m/n} = \sqrt[n]{x^m}$$

Important Notes:

- Radicals under even roots cannot be negative in real numbers.
- Simplify completely and combine like terms whenever possible.

8 Geometry Connections

8.1 Distance and Midpoint Formulas

Pythagorean Theorem: In a right triangle with legs of length a and b and hypotenuse c:

$$a^2 + b^2 = c^2$$

This tells us that the square of the hypotenuse (the longest side) equals the sum of the squares of the other two sides.

Distance Formula: The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the hypotenuse of a right triangle formed by the horizontal and vertical distances:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example: Find the distance between $A(1, 2)$ and $B(4, 6)$.

$$\begin{aligned} \text{Horizontal distance} &= |x_2 - x_1| = |4 - 1| = 3 \\ \text{Vertical distance} &= |y_2 - y_1| = |6 - 2| = 4 \\ \text{Distance} &= \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5 \end{aligned}$$

Midpoint Formula: The point exactly halfway between $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

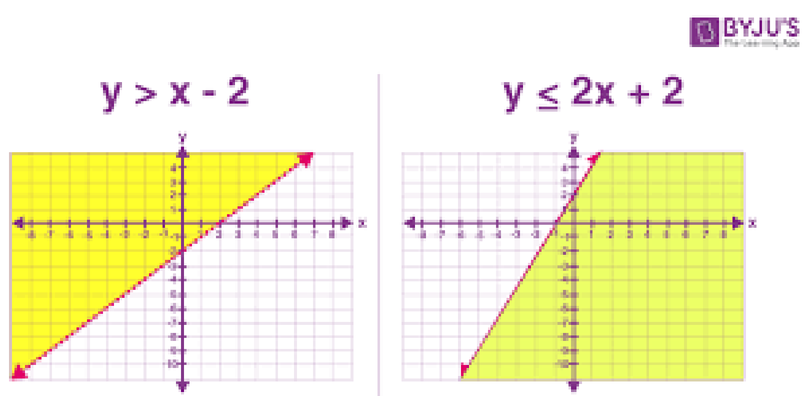
Example: Find the midpoint of $A(1, 2)$ and $B(4, 6)$:

$$M = \left(\frac{1 + 4}{2}, \frac{2 + 6}{2} \right) = (2.5, 4)$$

8.2 Linear Inequalities in Two Variables

Graphing Steps:

1. Rewrite inequality in slope-intercept form ($y = mx + b$).
2. Draw the boundary line: solid for $\leq$ or $\geq$, dashed for $<$ or $>$.
3. Shade the half-plane that satisfies the inequality.



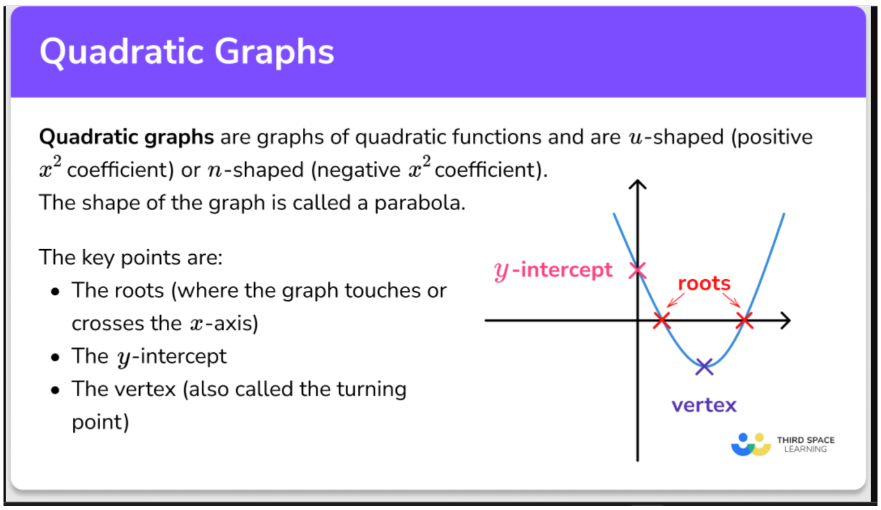
8.3 Quadratic Graphs and Roots

Quadratic equations: $y = ax^2 + bx + c$.

X-Intercepts: Solve $ax^2 + bx + c = 0$ to find the points where the parabola crosses the x-axis.

Example: Solve $y = x^2 - 5x + 6$:

$$x^2 - 5x + 6 = 0 \implies (x-2)(x-3) = 0 \implies x = 2, 3$$



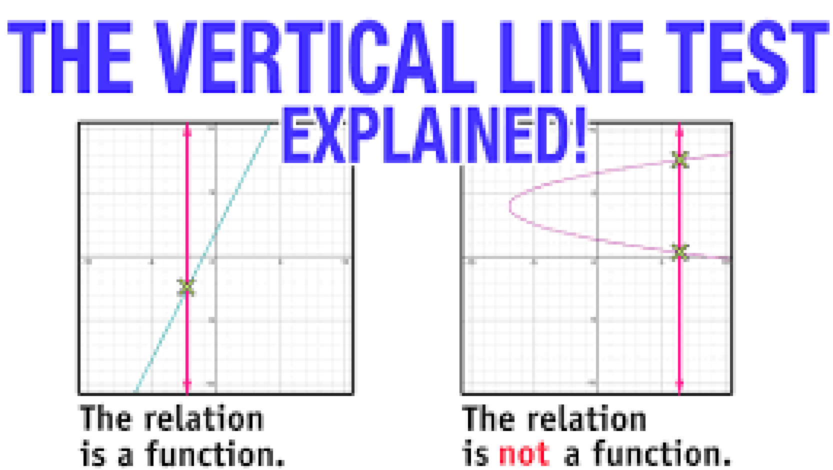
9 Functions and Relations

9.1 Definition of a Function

A **function** is a rule that assigns **exactly one output** for each input.

Vertical Line Test: - If a vertical line intersects the graph more than once, it is **not** a function. - If every vertical line intersects at most once, it **is** a function.

Example: - $f(x) = x^2$ is a function (every input x has exactly one output y). - $y^2 = x$ is **not** a function (for $x = 4$, $y = 2$ or $y = -2$, two outputs for one input).



9.2 Function Notation

- $f(x)$ means the output of function f for input x. - Think of $f(x)$ as "the result of x passed through f."

Example:

$$f(x) = 2x + 3$$

- $f(1) = 2(1) + 3 = 5$ - $f(4) = 2(4) + 3 = 11$

9.3 Representing Functions

Functions can be shown in different ways:

- **Ordered pairs:** $(x, f(x))$, e.g., $(1, 5), (2, 7), (3, 9)$
- **Tables:** list input and output values
- **Graphs:** plot points $(x, f(x))$ and connect
- **Mapping diagrams:** draw arrows from each input to its output

Example Table:

x	$f(x) = x + 2$
0	2
1	3
2	4
3	5

9.4 Domain and Range

- **Domain:** All possible input values (x-values). - **Range:** All possible output values (y-values).

Example: $f(x) = \sqrt{x - 1}$ - Domain: $x - 1 \geq 0 \implies x \geq 1$ - Range: $f(x) \geq 0 \implies y \geq 0$

9.5 Applications and Examples

Functions appear in real-life problems:

1. Direct Variation: - Formula: $y = kx$, where k is a constant of variation. - Step 1: Solve for k if you know a pair (x, y). - Step 2: Use k to find unknown values.

Example 1: Distance varies directly with time at constant speed. - A car travels 120 miles in 2 hours. Find k (speed).

$$y = kx \implies 120 = k \cdot 2 \implies k = \frac{120}{2} = 60$$

- Speed $k = 60$ miles per hour. - How far in 3 hours?

$$y = kx = 60 \cdot 3 = 180 \text{ miles}$$

Example 2: A recipe calls for 4 cups of flour to make 8 cookies. How much flour for 12 cookies?

$$y = kx, \quad y = \text{flour}, x = \text{cookies}$$

- Solve for k: $4 = k \cdot 8 \implies k = 0.5$ (cups per cookie) - For 12 cookies: $y = 0.5 \cdot 12 = 6$ cups

2. Inverse Variation: - Formula: $y = k/x$, where k is a constant of variation. - Step 1: Solve for k if you know a pair (x, y). - Step 2: Use k to find unknown values.

Example 1: Workers and time for a job: 6 workers complete a task in 8 days. How long for 12 workers?

$$y = \frac{k}{x} \implies 8 = \frac{k}{6} \implies k = 48$$

- Now for 12 workers: $y = \frac{48}{12} = 4$ days

Example 2: A machine can produce 300 items in 5 hours. How many hours for 600 items?

$$y = \frac{k}{x}, \quad x = \text{production rate}, y = \text{time}$$

- Solve for k: $5 = \frac{k}{300} \implies k = 1500$ - For 600 items: $y = \frac{1500}{600} = 2.5$ hours

Summary Steps: 1. Identify if it is direct or inverse variation. 2. Use the known pair to solve for k. 3. Substitute k and the new value to find the unknown.

3. Sequences as Functions: - Arithmetic sequence: $a_n = a_1 + (n-1)d$ - Example: $a_1 = 3, d = 2,\ a_5 = 3 + 4 \cdot 2 = 11$ - Geometric sequence: $a_n = a_1 \cdot r^{n-1}$ - Example: $a_1 = 2, r = 3,\ a_4 = 2 \cdot 3^3 = 54$